Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 3, N. 1, 2015.
Trabalho apresentado no XXXV CNMAC, Natal-RN, 2014.

Eigenvalues of dot-product kernels on the sphere

Douglas Azevedo
DAMAT- UTFPR
Campus Cornélio Procópio
E-mail: douglasa@utfpr.edu.br

Valdir A. Menegatto
Departamento de Matemática, ICMC, USP
E-mail: menegatt@icmc.usp.br.

Resumo: *We obtain explicit formulas for the eigenvalues of integral operators generated by continuous dot product kernels defined on the sphere via the usual gamma function. Using them, we present both, a procedure to describe sharp bounds for the eigenvalues and their asymptotic behavior near 0. We illustrate our results with examples, among them the integral operator generated by a Gaussian kernel .*

Palavras-chave: *sphere, integral operators, eigenvalue estimates, dot product kernels, Gaussian kernel.*

1 Introduction

Let S^m ($m \geq 2$) be the unit sphere in $\mathbb{R}^{m+1}$ endowed with its induced Lebesgue measure σ_m and write $L^2(S^m) := L^2(S^m, \sigma_m)$. In this work we deal with (compact) integral operators $\mathcal{K} : L^2(S^m) \to L^2(S^m)$ of the form

$$\mathcal{K}(f)(x) = \int_{S^m} \left(\sum_{n=0}^{\infty} b_n (x \cdot y)^n \right) f(y)\, d\sigma_m(y), \quad x \in S^m, \quad f \in L^2(S^m), \tag{1}$$

in which $\{b_n\}$ is an absolutely summable sequence of complex numbers. The symbol $\cdot$ stands for the usual inner product of $\mathbb{R}^{m+1}$. Kernels of the form

$$K(x, y) = \sum_{n=0}^{\infty} b_n (x \cdot y)^n, \quad x, y \in S^m, \tag{2}$$

with $\sum_{n=0}^{\infty} |b_n| < \infty$, will be called *dot product kernels* on S^m. They are bi-zonal in the sense that

$$K(x, y) = K'(x \cdot y), \quad x, y \in S^m,$$

for some convenient function $K' : [0, 1] \to \mathbb{C}$. In particular, every positive definite kernel on the Hilbert sphere of the usual space ℓ_2, as characterized in the early forties by Schoenberg ([15]) is a dot product kernel on every S^m. An eminent example from this category is the Gaussian kernel

$$\exp(-d\|x - y\|^2) = e^{-2d} \sum_{n=0}^{\infty} \frac{(2d)^n}{n!} (x \cdot y)^n, \quad x, y \in S^m, \quad d > 0,$$

a common entity in many branches of mathematics, including radial basis interpolation, learning theory, support vector machines, regularization networks and Gaussian processes ([9, 14, 18]). In the above formula, $\| \cdot \|$ stands for the usual norm in $\mathbb{R}^{m+1}$.

Kernels as those described in the previous paragraph belong to a larger class of kernels, namely, that of the power series kernels ([18]). Indeed, the multinomial theorem leads to the formula

$$\sum_{n=0}^{\infty} b_n (x \cdot y)^n = \sum_{\alpha \in \mathbb{Z}_+^{m+1}} a_\alpha x^\alpha y^\alpha, \quad x, y \in S^m,$$

where

$$a_\alpha = \frac{b_{|\alpha|} |\alpha|!}{\alpha!}, \quad \alpha = (\alpha_1, ..., \alpha_{m+1}) \in \mathbb{Z}_+^{m+1},$$

$|\alpha| = \sum_{i=1}^{m+1} \alpha_i$ and $\alpha! = \alpha_1!...\alpha_{m+1}!$ for all $\alpha \in \mathbb{Z}_+^{m+1}$.

The search for bounds of eigenvalues or singular values of integral operators is a classical and prolific topic in functional analysis ([6, 8, 13]) with applications in many areas of mathematics. Mercer's theorem provides a direct connection of the subject with smoothness properties of positive definite kernels, a topic of relative importance in reproducing kernel Hilbert space theory. In another front, the same bounds are useful in error estimates for approximation problems in learning theory ([4, 11, 16]).

In the spherical setting we are considering here, the eigenvalue analysis of a compact integral operator usually takes place in the case when the operator itself is positive and the generating kernel is smooth. Smoothness is usually defined by a closed assumption on the generating kernel itself or some of its derivatives but may also be described by a Hölder or Lipschitz type condition. The goal is then to deduce decay rates for the sequence of eigenvalues of the integral operator and try to reach optimality of the decay. We refer the reader to [3, 5, 10] and references therein for some recent results on that stream.

The focus in the present work is to deduce sharp bounds for the eigenvalues of the integral operator (1). They will be obtained through an explicit formula to compute such eigenvalues based on certain numerical series involving the usual gamma function. The formulas are described in Section 2 while the bounds are deduced in Section 3. These bounds can be seen as generalizations of those obtained in [11], in the case when the generating kernel is a Gaussian one. The results have a close connection with some previous information on eigenvalue decay of integral operators we have obtained in [2]. Finally, we like to think that the results have connections with those in [17], where the decay for Gegenbauer coefficients of the restriction of usual positive definite radial basis functions to S^m were considered. As an application, in Section 4, we recast the case in which the kernel (2) is a Gaussian kernel.

2 Computing the eigenvalues of the integral operator

In this section, we find a closed formula to compute the eigenvalues of the integral operator $\mathcal{K}$ introduced in (1). In order to do that we need to invoke a few facts from harmonic analysis on the sphere ([7, 12]).

For any function $F : [-1, 1] \to \mathbb{C}$ satisfying

$$\int_{-1}^{1} |F(t)|(1 - t^2)^{(m-2)/2} dt < \infty,$$

and any spherical harmonic Y_k of degree k in $m + 1$ dimensions, the Funk-Hecke formula ([7, p.98]) states that

$$\int_{S^m} F(x \cdot y) Y_k(y)\, d\sigma_m(y) = \lambda_k(F) Y_k(x), \quad x \in S^m,$$

in which

$$\lambda_k(F) = \sigma_{m-1} \int_{-1}^{1} F(t) P_k^m(t)(1 - t)^{(m-2)/2}\, dt,$$

P_k^m is the Legendre polynomial of degree k associated to the dimension m and σ_{m-1} is the surface area of S^{m-1}.

Rodrigues formula for the Legendre polynomials ([12, p.23]) reads as follows

$$\int_{-1}^{1} f(t)P_k^m(t)(1-t^2)^{(m-2)/2}\,dt = \frac{1}{2^k}\frac{\Gamma(m/2)}{\Gamma(k+m/2)}\int_{-1}^{1} f^{(k)}(t)(1-t^2)^{k+(m-2)/2}\,dt,$$

whenever f is a function of class C^k in $[-1,1]$. In particular,

$$\int_{-1}^{1} t^n P_k^m(t)(1-t^2)^{(m-2)/2}\,dt = \frac{1}{2^k}\frac{\Gamma(m/2)}{\Gamma(k+m/2)}\int_{-1}^{1}\frac{n!}{(n-k)!}t^{n-k}(1-t^2)^{k+(m-2)/2}\,dt, \quad n=0,1,\ldots.$$

If $n-k$ is odd, then $t^{n-k}(1-t^2)^{k+(m-2)/2}$ is an odd function of t and, consequently, the integral in the right-hand side of the above equality vanishes. If $n-k$ is even and nonnegative, then

$$\begin{aligned}\int_{-1}^{1} t^{n-k}P_k^m(t)(1-t^2)^{k+(m-2)/2}\,dt &= \frac{1}{2}\int_0^1 u^{(n-k-1)/2}(1-u)^{k+(m-2)/2}\,du\\ &= \frac{1}{2}B((n-k+1)/2, k+m/2),\end{aligned}$$

in which B is the usual beta function given by the formula

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \operatorname{Re} x, \operatorname{Re} y \in (0,\infty).$$

Here, Γ stands for the usual gamma function.

Below, we will write $\mathcal{H}_k^{m+1}$ to denote the space of all spherical harmonics of degree k in $m+1$ variables with respect to the inner product of $L^2(S^m)$. An orthonormal basis of $\mathcal{H}_k^{m+1}$ will be written as $\{Y_{k,l} : k=0,1,2,...; j=1,...,N(k,m+1)\}$.

Theorem *Let $\mathcal{K}$ be as in (1). Then each $Y_{k,j}$ is an eigenfunction of $\mathcal{K}$ with associated eigenvalue given by the formula*

$$\lambda_k(\mathcal{K}) = \frac{\sigma_{m-1}\Gamma(m/2)}{2^{k+1}}\sum_{s=0}^{\infty} b_{2s+k}\frac{(2s+k)!}{(2s)!}\frac{\Gamma(s+1/2)}{\Gamma(s+k+(m+1)/2)}, \quad k\in\mathbb{Z}_+.$$

Proof Let us fix $Y_{k,j}\in\mathcal{H}_k^{m+1}$. The summability of $\{b_n\}$ and the Funk-Hecke formula ensure that

$$\mathcal{K}(Y_{k,j}) = \left(\sigma_{m-1}\sum_{n=0}^{\infty} b_n\int_{-1}^{1} t^n P_k^m(t)(1-t^2)^{(m-2)/2}\,dt\right)Y_{k,j}.$$

In particular, $Y_{k,j}$ is an eigenfunction of $\mathcal{K}$ with corresponding eigenvalue

$$\lambda_k(\mathcal{K}) := \sigma_{m-1}\sum_{n=0}^{\infty} b_n\int_{-1}^{1} t^n P_k^m(t)(1-t^2)^{(m-2)/2}\,dt.$$

We observe that the integral appearing in the series above can be bounded by a number not depending on n so that the series is, indeed, convergent. Due to the comments preceding the theorem, it is now clear that

$$\lambda_k(\mathcal{K}) = \frac{\sigma_{m-1}\Gamma(m/2)}{2^{k+1}}\sum_{s=0}^{\infty} b_{2s+k}\frac{(2s+k)!}{(2s)!}\frac{\Gamma(s+1/2)}{\Gamma(s+k+(m+1)/2)}.$$

DOI: 10.5540/03.2015.003.01.0039

This concludes the proof of the theorem.

Clearly, the previous theorem implies that $\lambda_k(\mathcal{K})$ is an eigenvalue of $\mathcal{K}$ of multiplicity $N(k, m+1)$. Before we proceed with our intended target, that is, the bounding of the individual eigenvalues, we will detach a consequence of the previous theorem. It asserts that the nonincreasingness of the sequence $\{b_n\}$ suffices for the sequence of eigenvalues of $\mathcal{K}$ to match the outcome of the spectral theorem for compact operators on a Hilbert space.

Corolary *Let $\mathcal{K}$ be as in (1). If $\{b_n\}$ is a non increasing sequence of real numbers then $\{\lambda_k(\mathcal{K})\}$ decreases.*

Proof *For a fixed k, it is easily seen that*

$$\frac{(2s+k)!}{\Gamma(s+k+(m+1)/2)} > \frac{(2s+k+1)!}{\Gamma(s+k+1+(m+1)/2)}, \quad s = 0, 1, \ldots$$

Using this inequality and the non increasingness of $\{b_n\}$ to estimate $\lambda_{k+1}(\mathcal{K})$ leads to the inequality $\lambda_k(\mathcal{K}) \geq \lambda_{k+1}(\mathcal{K})$, $k = 0, 1, \ldots$, with equality only when all the b_n are 0.

We observe that the sequence mentioned in the previous result is not the sequence of eigenvalues of $\mathcal{K}$, since it does not take into account the multiplicities. However, the property obviously holds for the sequence of eigenvalues too.

At this point, it is interesting to observe that in some cases, Mercer's theorem ensures not only the non increasingness of the sequence of eigenvalues of $\mathcal{K}$ but also their summability (including multiplicities).

The closing result of the section describes a convenient lower bound for $\lambda_k(\mathcal{K})$ to be invoked later.

Corolary *Let $\mathcal{K}$ be as in (1). If $\{b_k\}$ is a sequence of nonnegative numbers, then there exists a positive constant C, depending upon m only, so that*

$$\lambda_k(\mathcal{K}) \geq C \frac{b_k}{2^{k+1}} \frac{k!}{\Gamma(k+(m+1)/2)}, \quad k \in \mathbb{Z}_+.$$

Proof If each b_k is nonnegative, then the series in the expression for $\lambda_k(\mathcal{K})$ obtained in the previous theorem is composed of nonnegative terms. As so, we can replace the whole series with its first term ($s = 0$) to deduce that

$$\lambda_k(\mathcal{K}) \geq \sigma_{m-1} \Gamma(1/2) \Gamma(m/2) \frac{b_k}{2^{k+1}} \frac{k!}{\Gamma(k+(m+1)/2)}, \quad k \in \mathbb{Z}_+.$$

The result follows.

3 Estimates for the eigenvalues

In this section, we intend to deduce upper bounds for $\lambda_k(\mathcal{K})$. Due to the nontrivial nature of the formula deduced in the previous section, sharp bounds will demand additional information on the sequence $\{b_n\}$. Below, we introduce notation and the additional requirement (decay) we will adopt for the sequence $\{b_n\}$.

If $\{a_n\}$ and $\{c_n\}$ are sequences of positive real numbers, as usual, $a_n = O(c_n)$ as $n \to \infty$, will mean that $\{a_n c_n^{-1}\}$ is bounded. On the other hand, $a_n \sim c_n$, as $n \to \infty$, will mean that $\lim_{n\to\infty} a_n/c_n = 1$, while $a_n \asymp c_n$, as $n \to \infty$, will mean that $a_n = O(c_n)$ and $c_n = O(a_n)$, as $n \to \infty$.

The lemma below describes an estimation for the series in the expression that defines $\lambda_k(\mathcal{K})$ when a decay for $\{b_n\}$ is available. We recall that

$$\theta_{k,s} := b_{2s+k} \frac{(2s+k)!}{(2s)!} \frac{\Gamma(s+1/2)}{\Gamma(s+k+(m+1)/2)}, \quad k, s \in \mathbb{Z}_+.$$

Lemma Let $\mathcal{K}$ be as in (1) and $\delta > 1/2$. If

$$\frac{|b_n|}{|b_{n-1}|} = O(n^{-\delta}), \quad (n \to \infty), \tag{3}$$

then there exists a positive real number γ so that

$$\sum_{s=0}^{\infty} |\theta_{k,s}| \leq |\theta_{k,0}| \sum_{s=0}^{\infty} \frac{\gamma^{2s}}{4^{\delta s}(s!)^{2\delta}}, \quad k \in \mathbb{Z}_+.$$

The proposition below is a crude estimation for the eigenvalues of $\mathcal{K}$ under the decay assumption (3).

Proposition *Let $\mathcal{K}$ be as in (1) and $\delta > 1/2$. If (3) holds, then*

$$|\lambda_k(\mathcal{K})| \leq \sigma_{m-1} \frac{\Gamma(1/2)\Gamma(m/2)}{\Gamma(k+(m+1)/2)} \frac{k!}{2^{k+1}} \left(\sum_{s=0}^{\infty} \frac{\gamma^{2s}}{4^{\delta s}(s!)^{2\delta}} \right) |b_k|, \quad k \in \mathbb{Z}_+.$$

Theorem *Let $\mathcal{K}$ be as in (1) and $\delta > 1/2$. If (3) holds and $\{b_n\}$ is a sequence of nonnegative numbers, then*

$$\lambda_k(\mathcal{K}) \asymp \frac{b_k}{2^{k+1} k^{(m-1)/2}}, \quad (k \to \infty).$$

Proof. We know from the previous results that there exists a positive constant C, depending upon m only, so that

$$C \frac{k!}{\Gamma(k+(m+1)/2)} \frac{b_k}{2^{k+1}} \leq \lambda_k(\mathcal{K}) \leq C \left(\sum_{s=0}^{\infty} \frac{\gamma^{2s}}{4^{\delta s}(s!)^{2\delta}} \right) \frac{k!}{\Gamma(k+(m+1)/2)} \frac{b_k}{2^{k+1}},$$

for all $k \in \mathbb{Z}_+$. An application of the Stirling's approximation formula

$$\Gamma(x) \sim \sqrt{2\pi}\, x^{x-1/2} e^{-x}, \quad (\operatorname{Re} x \to \infty),$$

yield the asymptotic formula

$$\frac{k!}{\Gamma(k+(m+1)/2)} \sim e^{(m+1)/2} \left(1 + \frac{-(m+1)/2}{k+(m+1)/2}\right)^k \frac{k^{1/2}}{(k+(m+1)/2)^{m/2}},$$

as $k \to \infty$. Since $k \sim k + (m+1)/2$ as $k \to \infty$, it follows that

$$\frac{k!}{\Gamma(k+(m+1)/2)} \sim \frac{1}{k^{(m-1)/2}}, \quad (k \to \infty).$$

The asymptotic information in the statement of the theorem follows.

Example (Gaussian kernel) *For $r \in (0, \infty)$ fixed, consider the kernel*

$$K(x,y) = e^{r/2} \exp\left(-\frac{||x-y||^2}{r} \right), \quad x, y \in S^m.$$

Since $\|x-y\|^2 = 2 - 2(x \cdot y)$, $x, y \in S^m$, the previous formula can be re-written as

$$K(x,y) = \sum_{n=0}^{\infty} \frac{2^n}{n! r^n} (x \cdot y)^n, \quad x, y \in S^m.$$

Since $b_n = 2^n r^{-n}/n! > 0$, $n = 0, 1, \ldots$, assumption (3) holds with $\delta = 1$. As so, the previous Theorem implies that

$$\lambda_k(\mathcal{K}) \asymp \frac{1}{2 r^k\, k! k^{(m-1)/2}}, \quad (k \to \infty),$$

while Stirling's approximation formula leads to

$$\lambda_k(\mathcal{K}) \asymp \frac{(e/r)^k}{k^{k+m/2}}, \quad (k \to \infty).$$

The estimates obtained above match those derived in Theorem 2 of [11].

DOI: 10.5540/03.2015.003.01.0039

References

[1] D. Azevedo, V.A. Menegatto, Sharp estimates for eigenvalues of integral operators generated by dot product kernels on the sphere, *J. Approx. Theory*, 177 (2014) 57-68.

[2] D. Azevedo, V.A. Menegatto, Eigenvalue decay of integral operators generated by power series-like kernels. *Math. Inequal. Appl.*, to apear.

[3] Castro, M. H.; Menegatto, V. A., Eigenvalue decay of positive integral operators on the sphere. *Math. Comp.* 81 (2012), no. 280, 2303-2317.

[4] De Vito, E.; Caponnetto, A.; Rosasco, L., Model selection for regularized least-squares algorithm in learning theory. *Found. Comput. Math.* 5 (2005), no. 1, 59-85.

[5] Ferreira, J. C.; Menegatto, V. A.; Peron, A. P., Integral operators on the sphere generated by positive definite smooth kernels. *J. Complexity* 24 (2008), no. 5-6, 632-647.

[6] Grothendieck, A., La théorie de Fredholm. *Bull. Soc. Math. France* 84 (1956), 319-384.

[7] Groemer, H., Geometric applications of Fourier series and spherical harmonics. Encyclopedia of Mathematics and its Applications, 61. Cambridge University Press, Cambridge, 1996.

[8] Hille, E.; Tamarkin, J. D., On the characteristic values of linear integral equations. *Acta Math.* 57 (1931), no. 1, 1-76.

[9] Lu, F.; Sun, H., Positive definite dot product kernels in learning theory. *Adv. Comput. Math.* 22 (2005), 181-198.

[10] Menegatto, V. A.; Oliveira, C. P., Eigenvalue and singular value estimates for integral operators: a unifying approach. *Math. Nachr*, 258 (2012), no. 17-18, 2222-2232.

[11] Minh, H.Q.; Niyogi, P.; Yao, Y., Mercer's theorem, feature maps, and smoothing. In: Proceedings of 19th Annual Conference on Learning Theory, Pittsburgh, June 2006. Springer, Berlin.

[12] Müller, C., Analysis of Spherical Symmetries in Euclidean Spaces. Applied Mathematical Sciences 129, Springer, New York, 1997.

[13] Pietsch, A., Nuclear locally convex spaces, Ergebnisse Der Mathematick und Ihrer Grensgebiete, Volume 66, Springer-Verlag, New York/Heidleberg/Berlin, 1972.

[14] Schaback, R., Multivariate interpolation by polynomials and radial basis functions. *Constr. Approx.* 21 (2005), 293-317.

[15] Schoenberg, I. J., Positive definite functions on spheres. *Duke Math. J.* 9 (1942), 96-108.

[16] Smale, S.; Zhou, Ding-Xuan, Learning theory estimates via integral operators and their approximations, *Constr. Approx.*, 26 (2007), 153–172.

[17] zu Castell, W.; Filbir, F., Radial basis functions and corresponding zonal series expansions on the sphere. *J. Approx. Theory* 134 (2005), 65-79.

[18] Zwicknagl, B., Power series kernels. *Constr. Approx.* 29 (2009), no. 1, 61-84.

DOI: 10.5540/03.2015.003.01.0039