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Forward Combustion Fronts as Traveling Waves Through a Porous Medium

Jesus C. da Mota
Instituto de Matemática e Estatística-IME, UFG
74001-970, Goiânia, GO
jesus@ufg.br

Aparecido J. de Souza
Departamento de Matemática-DME, UFCG
58429-970, Campina Grande, PB
cido@dmat.ufcg.edu.br

Abstract: *Traveling wave combustion for a two-phase flow model that represents combustion of oil with oxygen or air in a porous medium was discussed in [1]. The discussion was limited to the case where the speed of the particles ahead of the combustion front is less than the fractional oil heat capacity. In this work we discuss the other case where the speed of the particles ahead of the combustion front is greater than the fractional oil heat capacity.*

Traveling waves combustion are represented by connecting orbits of a system of three ordinary differential equations that approach the nonhyperbolic equilibrium along its stable manifold and not along its center direction. For each set of physical parameters, we prove the existence of two distinct combustion front, with different combustion temperatures and speeds, unlike the case discussed in [1], where there is only one. The proofs use geometric singular perturbation theory.

Key words: *Traveling Waves, Combustion, Porous Medium.*

1 Introduction

The model describe the one-dimensional flow of gas and oil through a porous medium taking into account a combustion reaction into the medium. The flow is described by state quantities depending on $x \in \mathbb{R}$ and $t \geq 0$, the space and time coordinates. They are denoted as follows: the gas saturation is $s = s(x,t)$ (the oil saturation is $1 - s$); the absolute temperature, which is assumed to be the same for gas, oil, and rock at each (x,t), is $\theta = \theta(x,t)$; and the volume fraction of burned gas is $\epsilon = \epsilon(x,t)$.

Using balance equations, after some usual simplifications (see [1]), the model reduces to the following system of three reaction-convection-diffusion equations,

$$\frac{\partial s}{\partial t} + \frac{\partial f}{\partial x} = -\frac{\partial}{\partial x}\left(h\frac{\partial s}{\partial x}\right), \tag{1}$$

$$\tfrac{\partial}{\partial t}((\alpha - s)\theta - \eta\epsilon s) + \tfrac{\partial}{\partial x}((\beta - f)\theta - \eta\epsilon f) = \tfrac{\partial}{\partial x}\left((\theta + \eta\epsilon)h\tfrac{\partial s}{\partial x}\right) + \tfrac{\partial}{\partial x}\left(\gamma\tfrac{\partial \theta}{\partial x}\right), \tag{2}$$

$$\frac{\partial}{\partial t}(\epsilon s) + \frac{\partial}{\partial x}(\epsilon f) = -\frac{\partial}{\partial x}\left(\epsilon h\frac{\partial s}{\partial x}\right) + \zeta s q, \tag{3}$$

where α, β, γ, η, and ζ are nonnegative constants, depending on the physical properties of the porous medium and fluids.

Typical expressions for f and q are taken as in [2].

$$f(s,\theta) = \frac{s^2}{s^2 + (0.1+\theta)(1-s)^2}, \tag{4}$$

$$q(\epsilon,\theta) = (1-\epsilon)A_r e^{-\frac{E}{\theta-\theta_0}}, \quad \text{if } \theta > \theta_0 \text{ or } 0, \text{ if } \theta \leq \theta_0, \tag{5}$$

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where θ_0 is the ignition temperature. Regarding the function h, in our analysis we only use that it has negative values.

The following quotient called *fractional oil heat capacity* plays an important role in the analysis,

$$\varphi = \frac{\beta}{\alpha} = \frac{\rho_o C_o}{\rho_o C_o + \rho_r C_r/\phi}. \tag{6}$$

It represents the fraction of the heat capacity of the oil as compared to the total heat capacity.

In what follows the unburned state denoted by $W_0 = (s_0, \theta_0, \epsilon = 0)$ will always be fixed. It could be fixed such that $\frac{f_0}{s_0} < \varphi$, $\frac{f_0}{s_0} = \varphi$ or $\frac{f_0}{s_0} > \varphi$, where $f_0 = f(s_0, \theta_0)$ and $\frac{f_0}{s_0}$ represents the velocity of the fluid particles prior to passage of the combustion front.

Combustion fronts as traveling wave solutions of the system (1)–(3) was studied in [1] for the case $\frac{f_0}{s_0} < \varphi$. In this work we focus the case $\frac{f_0}{s_0} > \varphi$, which is more realistic, as will be seen in Section 2.

Our main result is given in Section 3. We prove in Theorem 2 the existence of two temperature values for which forward combustion fronts develop in contrast with the case $\frac{f_0}{s_0} < \varphi$, where only one temperature value exists.

2 Preliminary results

In this Section we recall some notation introduced in [1, 2, 3, 4] and the main results obtained in [1].

We define

$$\Omega = \{g : \mathbb{R} \to \mathbb{R} : g \in C^1, \lim_{\xi\to\pm\infty} g(\xi) \text{ exists, and } \lim_{\xi\to\pm\infty} \frac{dg}{d\xi} = 0\}.$$

Let $\Omega^n = \Omega \times \cdots \times \Omega$ (n times). Given $X = (g_1, \ldots, g_n)$ in Ω^n, let

$$X^{\pm} = \lim_{\xi\to\pm\infty} X(\xi) = \lim_{\xi\to\pm\infty} \big(g_1(\xi), \ldots, g_n(\xi)\big) = (g_1^{\pm}, \ldots, g_n^{\pm})\,. \tag{7}$$

A traveling wave of the system (1)–(3) with speed σ, connecting a state $W_L = (s_L, \theta_L, \epsilon_L)$ on the left to a state $W_R = (s_R, \theta_R, \epsilon_R)$ on the right, is a solution $W(\xi) = \big(s(\xi), \theta(\xi), \epsilon(\xi)\big)$ in Ω^3, with $\xi = x - \sigma t$, satisfying the boundary conditions

$$W^- = W_L \quad \textit{and} \quad W^+ = W_R\,. \tag{8}$$

A traveling wave that connect a burned state ($\epsilon = 1$) on the left to an unburned state ($\epsilon = 0$) on the right is called a combustion wave.

Let $W_0 = (s_0, \theta_0, 0)$ and $W_1 = (s_1, \theta_1, 1)$ be fixed states unburned and burned, respectively. A function $W(\xi) = \big(s(\xi), \theta(\xi), \epsilon(\xi)\big)$ is a traveling wave of system (1)–(3) with speed σ connecting W_1 on the left to W_0 on the right if and only if $W(\xi)$ is an orbit of the system of ordinary differential equations

$$\dot{s} = \frac{a + \sigma s - f(s,\theta)}{h(s,\theta)}, \tag{9}$$

$$\dot{\theta} = \frac{1}{\gamma}((\beta - \sigma\alpha - a)\theta - \eta a\epsilon + b), \tag{10}$$

$$\dot{\epsilon} = \frac{\zeta}{a} s q(\epsilon, \theta), \tag{11}$$

satisfying the boundary contitions

$$W^- = W_1 \quad \text{and} \quad W^+ = W_0, \tag{12}$$

where $a = f_0 - \sigma s_0$ and $b = (f_0 - \beta - \sigma(s_0 - \alpha))\theta_0$. The speed σ and the states W_0 and W_1 are related by the Rankine–Hugoniot condition

$$\sigma = \frac{f_1 - f_0}{s_1 - s_0} = \frac{(\beta - f_0)\theta_0 + \eta f_0 - (\beta - f_0)\theta_1}{(\alpha - s_0)\theta_0 + \eta s_0 - (\alpha - s_0)\theta_1}, \tag{13}$$

where $f_0 = f(s_0, \theta_0)$ and $f_1 = f(s_1, \theta_1)$.

We remark that, with the σ value defined by (13), the states W_0 and W_1 are equilibria of (9)-(11) and the orbit defining the combustion wave leaves W_1 and enters W_0.

For a fixed $W_0 = (s_0, \theta_0, 0)$ and $W_1(s_1, \theta_1, 1)$ varying on the plane $\epsilon = 1$ of the state space, equation (13) defines a curve containing possible states of being connected to W_0. Figures 2, 2 and 2, show this curve for cases $\frac{f_0}{s_0} < \varphi$, $\frac{f_0}{s_0} = \varphi$ and $\frac{f_0}{s_0} > \varphi$, respectively.

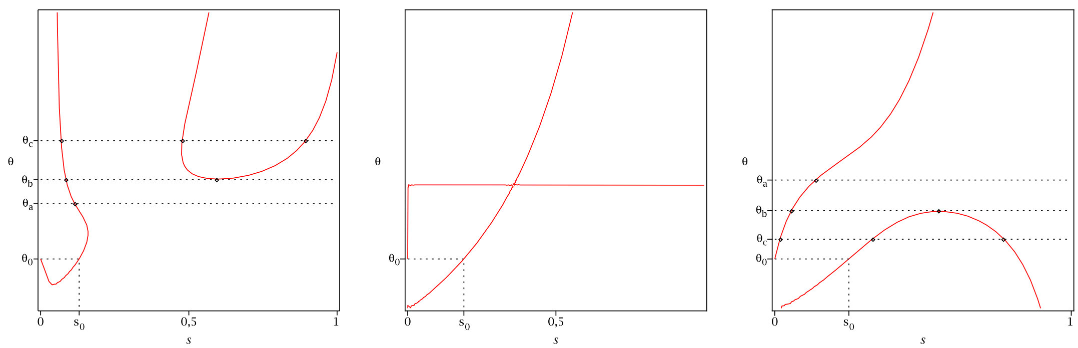

Figure 1: Curves containing possible states of being connected to W_0 by a combustion wave.

Figures 2 and 2 show three temperature values, denoted by θ_a, θ_b and θ_c, for which there are one, two or three possible burned states ($\epsilon = 1$) that can be connected to W_0 by a combustion wave. To find the correct burned states it is necessary to study the phase portrait of ODE system (9)–(11) depending on σ as a parameter. To do this, let us make the following change of variable in the temperature

$$T = \frac{\theta - \theta_0}{\theta_1 - \theta_0}, \tag{14}$$

that transforms the interval $\theta_0 \leq \theta \leq \theta_1$ to the interval $0 \leq T \leq 1$. The variable T will play the role of temperature. Thus, un unburned state now has the form $W_0 = (s_0, 0, 0)$ and a burned state has the form $W_1 = (s_1, 1, 1)$.

To simplify the notation, from now on we denote the specific temperature of combustion θ_1 only by θ.

In variables (s, T, ϵ) equations (9)–(12) (with θ_1 changed to θ and $\epsilon_1 = 1$) read

$$\dot{s} = X \equiv \frac{a + \sigma s - f(s, (\theta - \theta_0)T + \theta_0)}{h(s, (\theta - \theta_0)T + \theta_0)}, \tag{15}$$

$$\dot{T} = Y \equiv -\frac{b}{\gamma \theta_0}(T - \epsilon), \tag{16}$$

$$\dot{\epsilon} = Z \equiv \frac{\zeta A_r}{a} s(1 - \epsilon) e^{\frac{-E}{(\theta - \theta_0)T}}, \text{ if } T > 0, \text{ or } 0, \text{ if } T \leq 0, \tag{17}$$

satisfying

$$W^- = W_1 \quad \text{and} \quad W^+ = W_0. \tag{18}$$

The speed σ of the combustion front and consequently the constants a and b depend on the saturation s_0 and temperature θ_0 of the unburned state but, only on the temperature of combustion θ, since from (13)

$$\sigma(\theta) = \frac{A - B\theta}{C - D\theta} = \frac{B}{D} + \frac{\alpha \eta s_0 (\varphi - f_0/s_0)}{D^2(\theta - C/D)}, \tag{19}$$

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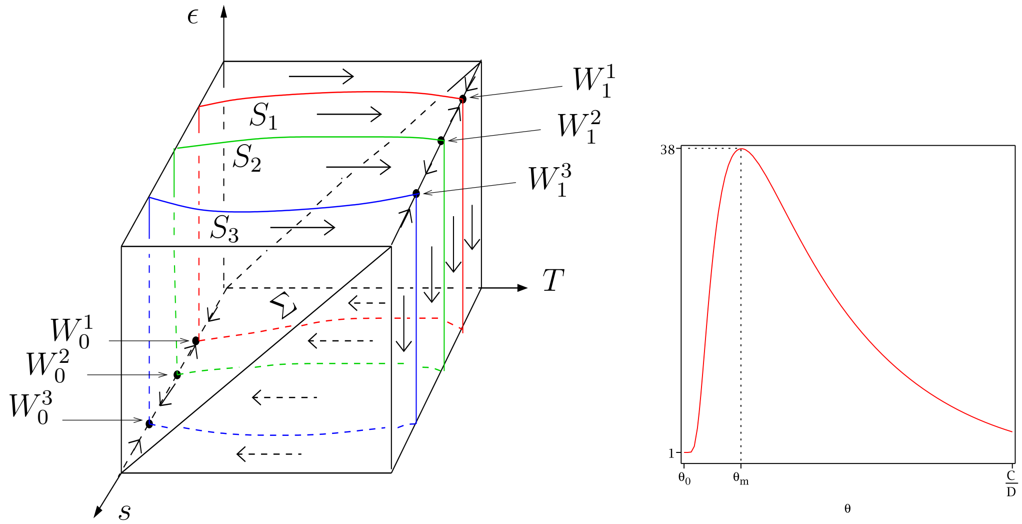
where

$$A = (\beta - f_0)\theta_0 + \eta f_0,\ B = \beta - f_0,\ C = (\alpha - s_0)\theta_0 + \eta s_0,\ \text{and } D = \alpha - s_0. \tag{20}$$

Considering $\theta = \theta_c$ as in Figure 2 or 2, system (15)–(17) has up to six equilibria, three on the plane $\epsilon = 1$, denoted by $W_1^1 = (s_1^1, 1, 1)$, $W_1^2 = (s_1^2, 1, 1)$ and $W_1^3 = (s_1^3, 1, 1)$, and three on the plane $\epsilon = 0$, denoted by $W_0^1 = (s_0^1, 0, 0)$, $W_0^2 = (s_0^2, 0, 0)$ and $W_0^3 = (s_0^3, 0, 0)$. These states are shown in Figure 2(a), where W_0^1 corresponds to W_0 previously fixed. Figure 2(a) also shows the surface branches where $\dot{s} = 0$, denoted by S_1, S_2 and S_3, and the plane where $\dot{T} = 0$ ($\epsilon = T$), denoted by Σ.



(a) (b)

Figure 2: (a) Equilibria $W_0^1, \cdots, W_1^3$ and (b) Slope $\frac{d\epsilon}{dT}$, as a function of θ, of the unstable manifold at V_1 of system (21)- (22).

A connecting orbit of an ODE from a hyperbolic equilibrium W_1 to a nonhyperbolic equilibrium W_0 is called strong if it lies in the stable manifold of W_0. If the connecting orbit corresponds to a traveling wave of a PDE, the traveling wave is also called strong.

Numerical simulations and physical reasons as remarked in [3] indicate that only the strong connections correspond to combustion fronts.

For the case where $f_0/s_0 < \varphi$ the main results obtained in [1] are summarized in the following.

Given an initial unburnt equilibrium W_0^1 with $f_0/s_0 < \varphi$, there exists two temperatures Θ_1 and Θ_2, $\Theta_1 < \Theta_2$, such that the equilibria W_1^1 and W_1^2 (with temperature θ) can be connected to W_0^1 by a traveling wave if and only if $C/D < \theta \leq \Theta_2$, where C and D are defined in (20). For θ within $(C/D, \Theta_1)$, the orbit enters W_0^1 tangent to the plane $\epsilon = T$ (this orbit is a stable separatrix of a central manifold of W_0), while for θ within $[\Theta_1, \Theta_2]$ the corresponding orbit enters W_0^1 tangent to the plane $\epsilon = 0$. Thus, this orbit is strong.

3 Main Results

Here we consider the case where $f_0/s_0 > \varphi$. It is more realistic because assuming that a forward combustion front exists the combustion temperature θ must be greater than the temperature θ_0 of the unburned state W_0. Therefore, from Eq. (13) it follows that $f_0/s_0 > \varphi$.

For each temperature θ, there are at most two burned states that can be connected to the unburned state W_0^1 by a combustion wave, which are W_1^1 and W_1^2.

To analyse these possible connections, we first consider combustion waves for the immobile oil model by freezing a positive value of the saturation, which we denote by r, $(0 < r \leq 1)$. In such case, system (15)–(17) reduces to

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$$\dot{T} = Y(T,\epsilon;\,\theta) \equiv -\frac{b(\theta)}{\gamma\theta_0}(T-\epsilon), \tag{21}$$

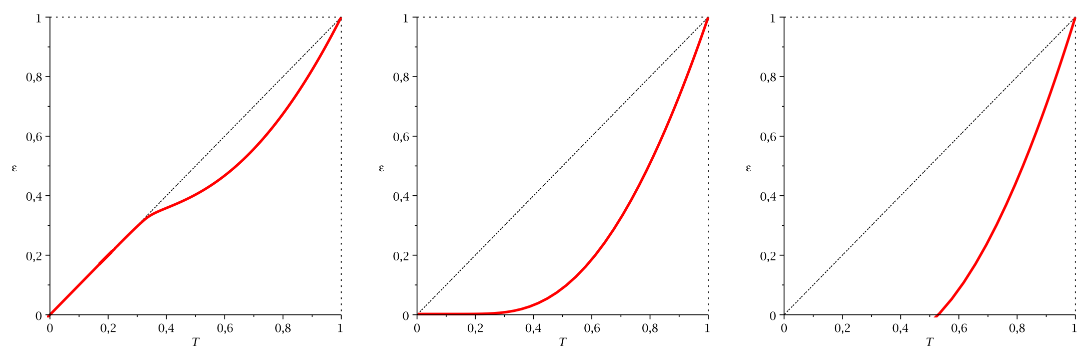
$$\dot{\epsilon} = Z(T,\epsilon;\,r,\theta) \equiv \frac{\chi}{a(\theta)} r(1-\epsilon)e^{\frac{-E}{(\theta-\theta_0)T}} \quad \text{if} \quad \theta > \theta_0, \quad \text{or} \quad 0 \quad \text{if} \quad 0 \le \theta \le \theta_0. \tag{22}$$

This system corresponds to the case of a highly viscous oil which can be considered immobile.

There are only two equilibria of (21)-(22), $V_0 = (0,0)$ corresponding to the unburned state and $V_1 = (1,1)$ corresponding to the burned state. These equilibria do not depend on the parameters r and θ.

For each r, there is a one-parameter family of traveling waves connecting V_1 to V_0. One end of this family corresponds to a strong connection (see Figure 3).

Let $r \in (0,\,1]$ be fixed. The slope $\frac{d\epsilon}{dT}$ of the unstable manifold at V_1, as a function of θ, has a unique maximum point $\theta_m = \theta_m(r) \in \left(\theta_0,\ \frac{C}{D}\right)$. Just look at the graph of $\frac{d\epsilon}{dT}$ as a function of θ, Figure 2.



(a) $\theta = 0.420$ (b) $\theta = 0.592$ (c) $\theta = 0.650$

Figure 3: Three possibilities for the unstable manifold of equilibrium $V_1 = (1,1)$.

In the next theorem the parameter r is fixed in the range $0 < r_0 \le r \le 1$.

Theorem 1. *Let $\theta_m = \theta_m(r)$, there are exactly two temperatures, $\theta_1^* = \theta_1^*(r)$ and $\theta_2^* = \theta_2^*(r)$, with $\theta_0 < \theta_1^* < \theta_m$ and $\theta_m < \theta_2^* < \frac{C}{D}$, such that the unstable manifolds of V_1, $\epsilon^u_{r\theta_1^*}$ and $\epsilon^u_{r\theta_2^*}$, enter in the equilibrium $V_0 = (0,0)$ tangent to the T-axis, that is, they are strong connections from V_1 to V_0.*

Proof. The proof is a consequence of continuity of (X,Y) flow with respect to parameter θ. See Figures 4(a) and 4(b), where $\tilde{\theta}$ and $\tilde{\tilde{\theta}}$ are temperature values near to θ_0 and to $\frac{C}{D}$, respectively. □

Let $\Delta = \{(T,\epsilon):\ 0 \le T \le 1;\ 0 \le \epsilon \le T\}$. For fixed parameters r, $0 < r \le 1$ and θ, $\theta_0 < \theta < \frac{C}{D}$, we denote by $\epsilon^s_{r\theta}$ the part of the stable manifold entering $V_0 = (0,0)$ which lies in the triangle Δ, possibly completed by a segment of the diagonal $\epsilon = T$.

We denote by S^r_θ the cylindrical surface $\epsilon = \epsilon^r_\theta(T)$, $0 \le s \le 1$, in the space (s,T,ϵ). We say that this surface is generated by $\epsilon = \epsilon^r_\theta(T)$. See Figures 5(a) and 5(b), where the surfaces S^r_θ are generated by the sets ϵ^r_θ.

We denote by A^r_θ the region formed by all points (s,T,ϵ) located above and on the surface S^r_θ, below the plane $\epsilon = T$, and with $0 \le s \le 1$. We also denote by B^r_θ the region located below and on the surface S^r_θ with $\epsilon \ge 0$, $0 \le T \le 1$, and $0 \le s \le 1$. These regions are indicated in Figures 5(a) and 5(b). We now return to the system (15)–(17). Let $\theta_1^* = \theta_1^*(1)$ and $\theta_2^* = \theta_2^*(1)$ be the temperatures as in Theorem 1, for $r = 1$. If θ lies in the range $\theta_0 < \theta < \theta_1^*$ or $\theta_2^* < \theta < C/D$, then the θ-orbit departing from W_1^1 reaches W_0^1 tangent to the plane $\epsilon = T$.

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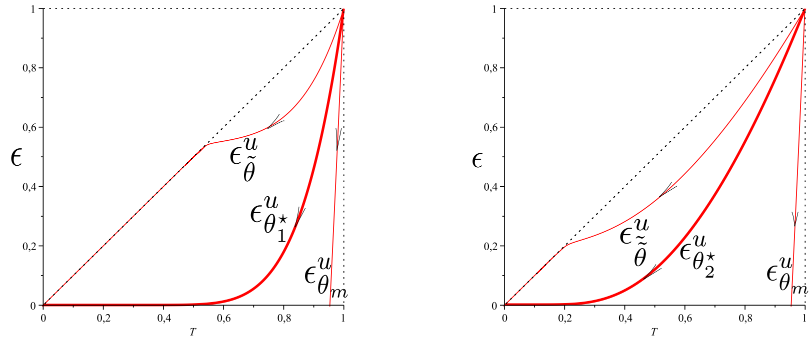
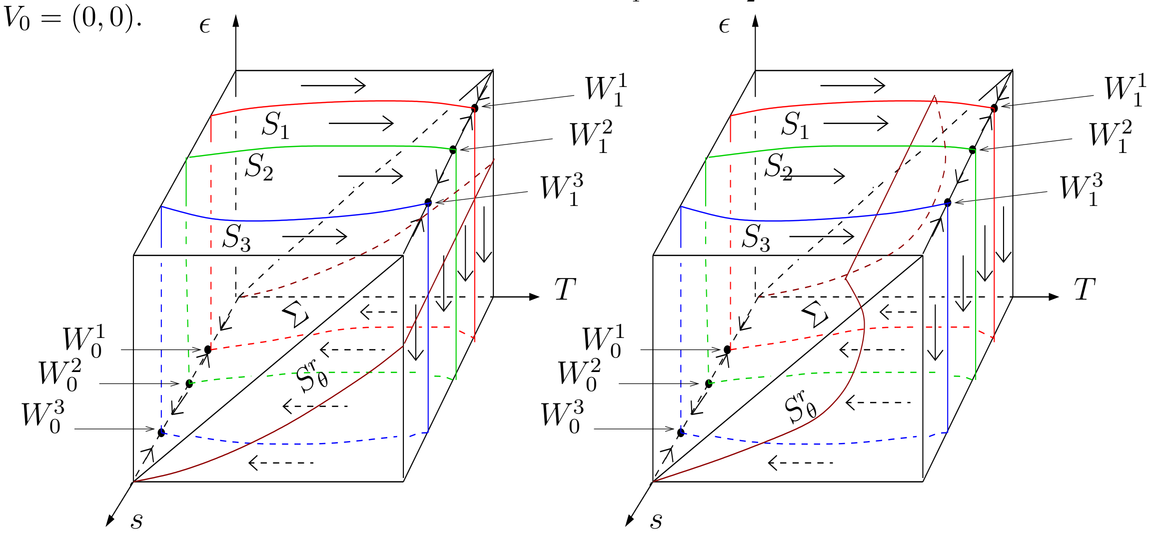


(a) $\tilde{\theta} < \theta_1^* < \theta_m$ (b) $\theta_m < \theta_2^* < \tilde{\tilde{\theta}}$

Figure 4: The unstable manifolds of $V_1 = (1,1)$, $\epsilon^u_{r\theta_1^*}$ and $\epsilon^u_{r\theta_2^*}$, are strong connections from V_1 to $V_0 = (0,0)$.



(a) (b)

Figure 5: (a) and (b): Cylindrical surfaces generated by the stable manifold $\epsilon^r_\theta(T)$ of $(0,0)$.

Note that as $\dot{s} > 0$ for $s = 0$ the θ-orbits enter the region of interest given by $0 \leq s \leq 1$, $0 \leq T \leq 1$ and $0 \leq \epsilon < 1$ along the plane $s = 0$. Based on this fact, let $\bar{r}$ a constant value of s, with $0 < \bar{r} < s_1^1$, such that $\dot{s}$ is positive along the plane $s = \bar{s}$.

Let $\theta_1^* = \theta_1^*(\bar{r})$ and $\theta_2^* = \theta_2^*(\bar{r})$ the temperature values as in Theorem 1. Given $\theta \in [\theta_1^*, \theta_2^*]$, the unstable manifolds of W_1^1 and of W_1^2 do not reach the equilibrium W_0^1, i.e., there are no orbits connecting W_1^1 or W_1^2 to W_0^1.

Now we define the following sets:

$$\Omega_1^* = \big\{\Theta \in \mathbb{R} \,|\, \text{for } \theta_0 < \theta \leq \Theta \text{ the the } \theta\text{-orbit leaving } W_1^1 \text{ (or } W_1^2) \\ \text{enters } W_0^1 \text{ tangent to the plane } \epsilon = T\big\},$$

$$\Omega_2^* = \big\{\Theta \in \mathbb{R} \,|\, \text{for } \Theta \leq \theta < \theta_m(\bar{r}) \text{ the } \theta\text{-orbit leaving } W_1^1 \text{ (or } W_1^2) \text{ do not enter } W_0^1\big\}.$$

The Ω_1^* set is non empty and it is upper bounded by $\theta_1^*(\bar{r})$. The Ω_2^* set is also non empty and it is lower bounded by $\theta_1^*(1)$.

Simillarly we define the sets:

$$\Omega_1^{**} = \big\{\Theta \in \mathbb{R} \,|\, \text{for } \theta_m(\bar{r}) \leq \theta < \Theta \text{ the } \theta\text{-orbit leaving } W_1^1 \text{ (or } W_1^2) \text{ do not enter } W_0^1\big\},$$

$$\Omega_2^{**} = \big\{\Theta \in \mathbb{R} \,|\, \text{for } \Theta \leq \theta < C/D \text{ the } \theta\text{-orbit leaving } W_1^1 \text{ (or } W_1^2) \\ \text{enters } W_0^1 \text{ tangent to the plane } \epsilon = T\big\}.$$

The Ω_1^{**} set is non empty, because $\theta_2^*(\bar{r}) \in \Omega_1^{**}$ and it is upper bounded by $\theta_2^*(1)$. The Ω_2^{**} set is also non empty, because $\theta_2^*(1) \in \Omega_2^{**}$ it is lower bounded by $\theta_2^*(\bar{r})$.

Thus, the following temperature values are well defined:

$$\Theta_1^* = sup\,\Omega_1^*, \quad \Theta_2^* = inf\,\Omega_2^*,$$
$$\Theta_1^{**} = sup\,\Omega_1^{**} \quad \text{and} \quad \Theta_2^{**} = inf\,\Omega_2^{**}\,,$$

and the following inequalities hold

$$\theta_0 < \theta_1^*(1) < \Theta_1^* \leq \Theta_2^* < \theta_1^*(\bar{r}) < \theta_m^*(\bar{r}) < \theta_2^*(\bar{r}) < \Theta_1^{**} \leq \Theta_2^{**} < \theta_2^*(1) < C/D\,. \tag{23}$$

By the continuity on the parameter θ, we have that Θ_1^*, Θ_2^*, Θ_1^{**} and Θ_2^{**} do not belong to the sets Ω_1^*, Ω_2^*, Ω_1^{**} and Ω_2^{**}, respectively.

As a consequence of the above results, we finally have,

Theorem 2. *If θ is such that $\Theta_1^* \leq \theta \leq \Theta_2^*$ or $\Theta_1^{**} \leq \theta \leq \Theta_2^{**}$, the θ-orbit living W_1^1 (or W_1^2) enters W_0^1 tangent to the plane $\epsilon = 0$, i.e., the θ-orbit is a strong connection from W_1^1 (or W_1^2) to W_0^1.*

In this work we prove the existence of two intervals for which strong connections, representing combustion fronts, can occur in contrast to the case $f_0/s_0 < \varphi$, where only one interval was obtained.

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