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A Model of Economic Growth in Two Spatial Dimensions

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Abstract. In this work we propose a two dimensional version of the spatial Solow-Swan economic growth model with capital-induced labor migration. As in the unidimensional model, the stability analysis results show that the capital-induced labor migration parameter must be greater than a critical value so that the model can generate static spatial agglomerations or spatio-temporal cycles. Numerical simulations show that a more intense capital-induced labor migration and returns to scale in the production process tend to increase the complexity of the two-dimensional spatio-temporal dynamical behavior of the economy. Moreover, in two spatial dimensions the static economic agglomerations generated by the model can assume one of two forms, spots or stripes.

Keywords. Spatial Economic Growth Model, Solow-Swan Model, Stability Analysis, Economic Agglomeration.

1 Introduction

The non-spatial Solow-Swan economic growth model is considered to be the benchmark growth model in the economic literature [1, 11, 12]. It considers that the time evolution of the aggregate capital and labor in an economy is governed by a system of two non-linear ordinary differential equations. The number of workers in the economy is postulated to grow exponentially through a Malthusian process at a constant rate, and it is usual to consider that the capital is reproduced through a Cobb-Douglas production function [1], which combines existing capital and workers in order to produce new capital, associated with an exponential depreciation of capital at a constant rate. An unidimensional spatial version of this model, considering that capital and workers show a diffusive movement through space, was proposed in [2, 3]. More recently, it was considered a workforce growing through a logistic process, and a capital-induced labor migration in the unidimensional spatial Solow-Swan model, such that the model was able to endogenously generate non-homogeneous spatial distributions of capital and labor in the steady state, as well as more complex spatio-temporal cycles [5–7].

The main objective of this paper is to extend the unidimensional model proposed in [5–7] to two spatial dimensions (section 2), to present the main stability results of the spatially homogeneous coexistence equilibrium (section 3), and to present some numerical simulations in two spatial dimensions using a finite-element numerical scheme (section 4). At last, we close with our final remarks in section 5.

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2 The Spatial Solow-Swan Model in Two Spatial Dimensions

The proposed model considers that the geographical space of a spatial economy is described by the region $\Omega \subset \mathbb{R}^2$, where each of its points $\mathbf{x} \in \Omega$ is considered a local economy. Besides, we consider that there is only one aggregated good in the economy, whose production uses two production factors, capital and labor, all considered in physical terms. In this way, each local economy is characterized by some capital density, $K(t, \mathbf{x}) \geq 0$, and labor density, $L(t, \mathbf{x}) \geq 0$. The capital evolution is governed by the following reaction-diffusion partial differential equation (PDE):

$$\frac{\partial K}{\partial t} = sA\left(K^{\phi} L^{1-\phi}\right)^{\gamma} - \delta K + d_K \Delta K, \tag{1}$$

where the first two terms in the right-hand side describe net investment, and the last one the spatial diffusion of capital to where it is scarcer, with Δ being the Laplacian operator in two dimensions in Cartesian coordinates. The following exogenous parameters are present in this equation: $\phi \in [0, 1]$ is the output elasticity of capital, $A > 0$ is the technological coefficient, $\gamma \in \left(0, \frac{1}{\phi}\right)$ gives the returns to scale[4] in the production process, $s \in [0, 1]$ is the saving rate, $\delta \in [0, 1]$ is the capital depreciation rate, and $d_K > 0$ is the diffusion coefficient of the capital.

The labor evolution is governed by the following reaction-diffusion-advection PDE:

$$\frac{\partial L}{\partial t} = aL - bL^2 + d_L \Delta L - \chi_L \nabla \cdot (L \nabla K). \tag{2}$$

In this equation, the reaction term describes a logistic organic growth for the workforce, the diffusion term the movement of labor to where it is scarcer, and the advection term the capital-induced labor migration, with workers moving to neighboring locations where there is more capital. Here $a > 0$ is the labor growth rate, $b > 0$ is a limiting factor, $\frac{a}{b} > 0$ is the local maximum labor capacity, $d_L > 0$ is the labor diffusion coefficient, and $\chi_L > 0$ is the capital-induced labor migration coefficient.

We will be interested in the stability of the spatially homogeneous coexistence equilibrium of the coupled system given by equations (1)-(2), which is given by $(K_\infty, L_\infty) = \left(\left[\frac{sA}{\delta}\left(\frac{a}{b}\right)^{(1-\phi)\gamma}\right]^{\frac{1}{1-\phi\gamma}}, \frac{a}{b}\right)$.

Rescaling the variables of the model in the following way: $K^* = \frac{K}{K_\infty}$, $L^* = \frac{L}{L_\infty}$, $t^* = at$, $x^* = \sqrt{\frac{a}{d_K}}x$, and dropping out the asterisks, we get the model in the simpler adimensional form:

$$\begin{aligned} \frac{\partial K}{\partial t} &= \beta\left[\left(K^{\phi} L^{1-\phi}\right)^{\gamma} - K\right] + \Delta K, \\ \frac{\partial L}{\partial t} &= L(1-L) + d\Delta L - \chi \nabla \cdot (L \nabla K), \end{aligned} \tag{3}$$

where $\beta = \frac{\delta}{a}$, $d = \frac{d_L}{d_K}$, and $\chi = \frac{\chi_L}{d_K}\left[\frac{sA}{\delta}\left(\frac{a}{b}\right)^{(1-\phi)\gamma}\right]^{\frac{1}{1-\phi\gamma}}$. In this form, the coexistence equilibrium of the model is normalized, $(K_\infty, L_\infty) = (1, 1)$.

Closing the model, we prescribe the initial distributions of capital and labor, $K(0, \mathbf{x}) = K_0(\mathbf{x}) \geq 0$, $L(0, \mathbf{x}) = L_0(\mathbf{x}) \geq 0$ for $\mathbf{x} \in \Omega$, and assume homogeneous Neumann boundary conditions for both capital and labor: $\frac{\partial K}{\partial \mathbf{n}}(t, \mathbf{x}) = \frac{\partial L}{\partial \mathbf{n}}(t, \mathbf{x}) = 0$, for $(t, \mathbf{x}) \in (0, \infty) \times \partial\Omega$, what economically means that the spatial economy Ω is considered to be an autarchy.

[4]If $\gamma < 1$ ($\gamma > 1$) we say that the production process has decreasing (increasing) returns to scale, i.e., doubling the quantity of all production factors, less (more) than doubles the total amount of production. Otherwise, if $\gamma = 1$ we have constant returns to scale, meaning that doubling the quantity of production factors exactly doubles the total amount of production.

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3 Stability Analysis Results

The stability analysis of the spatially homogeneous coexistence equilibrium of the 2D adimensional model (3) is very similar to the unidimensional model, whose details can be found in [5–7]. More general accounts can also be found in the references [4, 9, 10].

Linearizing the model (3) around the normalized equilibrium of coexistence, and considering solutions in the form of harmonic waves, $e^{\sigma t + i\mathbf{k}\cdot\mathbf{x}}$, we have that the coexistence equilibrium will be stable if $\Re\{\sigma\} < 0$ for all wavevectors $\mathbf{k} = (k_x, k_y)$. In the present case we can show that $\sigma = \sigma(q)$, $q = \|\mathbf{k}\|^2 > 0$, is solution of the following quadratic equation:

$$P(\sigma) = \sigma^2 + z(q)\sigma + w(q) = 0,$$

where:

$$\begin{aligned} z(q) &= (1+d)q + \beta(1-\phi\gamma) + 1, \\ w(q) &= dq^2 + \{1 + \beta[(1-\phi\gamma)d - (1-\phi)\gamma\chi]\}q + \beta(1-\phi\gamma). \end{aligned}$$

The following result, whose proof can be found in [7], give us a condition under which the spatially homogeneous equilibrium of coexistence in unstable. It is only when this equilibrium is unstable that we may expect the model to generate spatial economic agglomerations of labor and capital, as well as spatio-temporal cycles, features observed in real economies.

Proposition - Consider $\gamma \in \left(0, \frac{1}{\phi}\right)$. The spatially homogeneous coexistence equilibrium of the system (3), $(K_\infty, L_\infty) = (1, 1)$, is unstable if $\chi \geq \chi_c$, where χ_c is the critical value for the capital-induced labor migration coefficient, defined as:

$$\chi_c = \frac{1}{\beta(1-\phi)\gamma}\left(1 + \sqrt{d\beta(1-\phi\gamma)}\right)^2, \tag{4}$$

or, equivalently, if:

$$w(q) \leq 0 \text{ when } \chi \geq \chi_c. \tag{5}$$

Economically, this proposition shows that the model is able to generate economic agglomerations or spatio-temporal cycles only if the capital-induced labor migration is intense enough ($\chi \geq \chi_c$); otherwise, the economy always converges to a steady state spatially homogeneous. Moreover, greater returns to scale in the production function, i.e. a greater value of the parameter γ, implies in a smaller χ_c, what eases the spatial pattern formation.

4 Numerical Simulations

In the numerical simulations below we consider the square region $\Omega = [-5, 5] \times [-5, 5]$, and the following theoretical values for the model's parameters: $a = 0.02$, $b = 0.01$, $\delta = 0.05$, $s = 0.2$, $A = 1$, $\phi = 0.5$, $d_K = d_L = 1$. These dimensional parameter values imply that $\beta = 2.5$ and $d = 1$. We also consider the following returns to scale values: $\gamma = 0.6,\ 0.8,\ 1.0,\ 1.2,\ 1.4$, which imply in the following critical values for the capital-induced labor migration parameter: $\chi_c \approx 7.19,\ 4.95,\ 3,59,\ 2.67,\ 1.99$. The initial distributions of capital and labor considered were: $K(x, y) = L(x, y) = e^{-\frac{1}{25}(x^2+y^2)}$.

The numerical solutions are computed from the following standard Galerkin finite element approximation with the θ time scheme discretization:

$$
\begin{aligned}
\int_\Omega \frac{K^{n+1}-K^n}{\Delta t^n}\, v\, d\mathbf{x} &= \int_\Omega \left[\theta^n S(K^{n+1},L^{n+1}) + (1-\theta^n) S(K^n,L^n)\right] v\, d\mathbf{x} \\
&\quad - \int_\Omega \left[\theta^n \nabla K^{(n+1)} + (1-\theta^n)\nabla K^n\right]\cdot \nabla v\, d\mathbf{x}, \\
\int_\Omega \frac{L^{n+1}-L^n}{\Delta t^n}\, v\, d\mathbf{x} &= \int_\Omega \left[\theta^n Q(L^{n+1}) + (1-\theta^n) Q(L^n)\right]\, w\, d\mathbf{x} \\
&\quad - \int_\Omega d\left[\theta^n \nabla L^{n+1} + (1-\theta^n)\nabla L^n\right]\cdot \nabla w\, d\mathbf{x} \\
&\quad + \int_\Omega \chi\left[\theta^n L^{n+1}\nabla K^{n+1} + (1-\theta^n) L^n \nabla K^n\right]\cdot \nabla w\, d\mathbf{x},
\end{aligned}
\tag{6}
$$

where Δt^n denotes the time step at iteration the n-th iteration, $K^n \approx K(t^n,x)$, $L^n \approx L(t^n,x)$, the reaction terms are $S(K,L) = \beta\left[\left(K^\phi L^{1-\phi}\right)^\gamma - K\right]$ and $Q(L) = L(1-L)$. Test (v,w) and trial (K^n,L^n) functions are given in the finite element space of triangular P_1 elements built on a regular mesh of Ω. The non-linear finite element problem has been implemented in Python with the help of the FEniCSx project [8]. The simulations start with the implicit scheme $\theta^n = 1$ and change to semi-implicit scheme $\theta^n = 0.5$ after the fiftieth time iteration. Time step is tracking based on the convergence rate of the Newton solver.

In Figure 1 we present simulations for some representative scenarios, considering constant returns to scale, $\gamma = 1$, and varying the value of the capital-induced labor migration coefficient, χ. As we can see, a higher value of χ implies in more complex spatio-temporal behavior, since in this case there are more unstable modes. While for $\chi = 3 < \chi_c \approx 3,59$ the economy converges to a homogeneous steady state, for $\chi = 5$ and $\chi = 7$ we have static non-homogeneous steady states, with the coinciding capital and labor spatial agglomerations showing a stripe or spot (peak) pattern, respectively. For more intense capital-induced labor migration, $\chi = 8,\ 10,\ 12$, the economy develops irregular spatio-temporal cycles, with the peaks of capital and labor moving through the space. This behavior can also be seen in the temporal evolution of the economy's aggregate capital and labor, $K_a(t)$ and $L_a(t)$, which are presented in the third and fourth column of the figure. In general terms this behavior is similar to the unidimensional model presented in [5], but in two dimensions the spatial agglomerations can assume two different forms, stripes or spots, while in one dimension only the spot pattern is possible.

In Figure 2 we fix $\chi = 10$, and vary the returns to scale coefficient, γ. For decreasing returns to scale ($\gamma < 1$) we note that the economy converges to static steady states, while for constant ($\gamma = 1$), or increasing returns to scale ($\gamma > 1$), the emergence of spatio-temporal cycles are verified. This result shows the opposed behavior of the unidimensional model analyzed in [7]. We also note that higher returns to scale make the economy reach higher levels of capital, maintaining the total amount of labor approximately constant. In economic terms, this means that the labor productivity increases with increasing returns to scale, a result that goes in line with those presented in the unidimensional model [7].

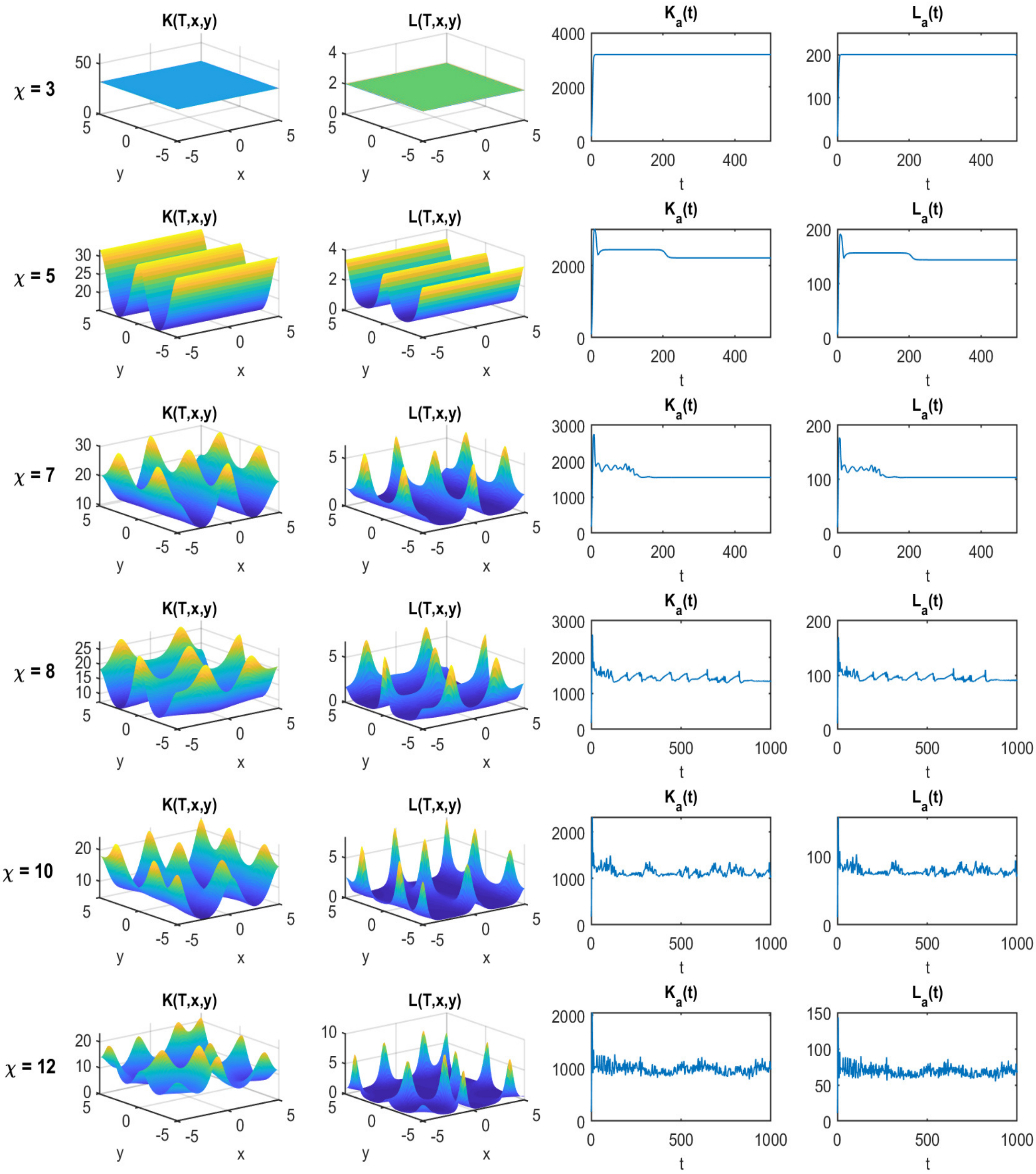

Figure 1: Numerical simulations results considering $\gamma = 1$ fixed, and $\chi = 3,\ 5,\ 7,\ 8,\ 10,\ 12$.

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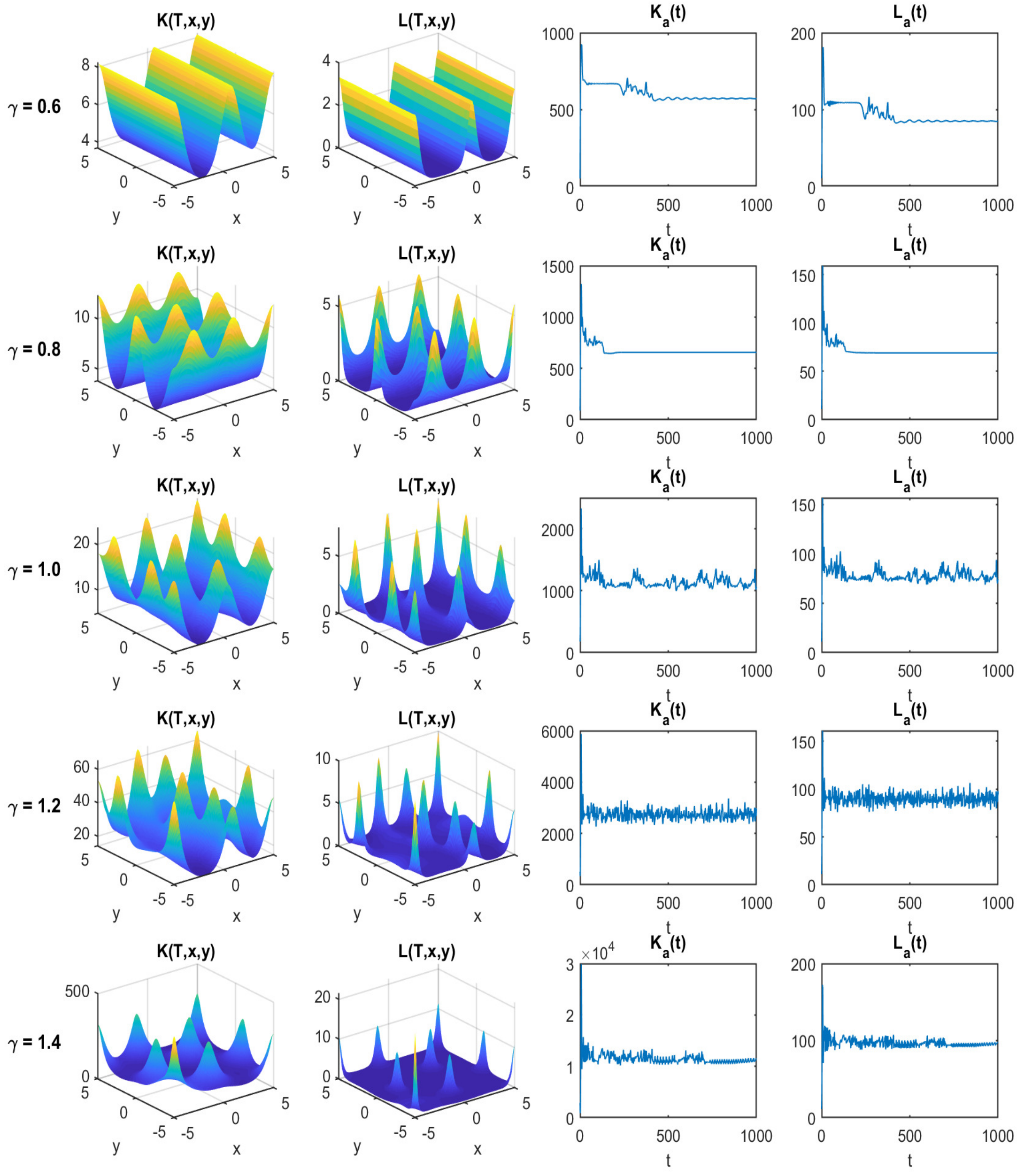

Figure 2: Numerical simulations results considering $\chi = 10$ fixed, and $\gamma = 0.6,\ 0.8,\ 1.0,\ 1.2,\ 1.4$.

5 Final Remarks

The proposed two dimensional generalization of the spatial Solow-Swan model presents, in general terms, a similar behavior to the unidimensional version of the model, however presenting two possible types of static spatial agglomerations, stripes or spots. In particular, the critical value χ_c remains the same in both models, and a more intense capital-induced labor migration still increases the complexity of the generated spatio-temporal dynamics. Further investigations may focus on explaining why, in the present generalization, a higher returns to scale, **ceteris paribus**, seems to generate more complex spatio-temporal dynamics, and also apply a non-linear stability analysis in order to verify if it is possible to predict the type of static agglomeration generated (spots or stripes) for given parameters of the model.

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