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Applications of Randers metric to track the paths through which the fire surrounds the power transmission lines

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Abstract. We utilize Randers geometry to model the propagation of fire waves, deriving the equations for paths surrounding power transmission lines in a flat terrain during a wildfire outbreak with varying wind patterns. These paths, known as strategic paths, can play a critical role in managing firefighting strategies. By identifying these paths, we can allocate firefighters and resources to specific locations instead of trying to cover a large area. This approach enables us to predict the path of the fire toward the transmission lines, thus reducing the time and cost of firefighting operations. We demonstrate the effectiveness of our approach using MATLAB by simulating two hypothetical wildfire scenarios.

Keywords. Randers metric, Fire front, Wave rays, Strategic path, Wildfire propagation.

1 Introduction

Wildfires damage human properties and assets, in particular, the transmission lines of the power system. In this work, we analyze wildfire propagation under the presence of space-varying wind on flat lands. We obtain equations of the paths through which the fire surrounds and calculate the time when the fire reaches the transmission lines. We call these paths the *strategic paths*.

One of the pioneering works concerning the study of fire propagation was done by Anderson [1]. Anderson in [1] presented the ellipse equations and applied Huygens' principle to find the fire fronts and the propagation model. Richard [11] derived a non-linear first-order system of equations for wildfire propagation with varying fuel. He introduced the simulators in which he used the solution to his first-order system and applied Huygens' principle to find wildfire propagation [12]. In [10], the author showed that the equations derived in [11] are the geodesics of a Finsler metric associated with the propagation. He also showed that using Finsler geodesics is a better approach to finding a more accurate propagation model and proved the validity of Huygens' principle for Finsler spaces of dimension two. Then, in [7], the authors showed the validity of Huygens' principle for Finsler spaces of every dimension and used the Finsler geodesics to find the wave propagation. In [5], by using the Randers geodesic, the equations of fire fronts and wave rays are derived. The authors in [**javaloyes2021general**] provide a more general model inspired by Matsumoto metric to take into account the slope of the terrain in a wildfire.

Finsler geometry is a strong tool to investigate the situations in anisotropic or nonhomogeneous media and has recently been used to study the wave propagation [5–7, 10]. Here, we use an essential class of Finsler metrics, Randers, to analyze wildfire behavior. The *strategic paths* are the geodesics of Randers metric associated with the propagation that obey some special conditions [5].

The main contribution of this work is the presentation of the equation of the strategic paths based on mathematical approaches. Discovering strategic paths is beneficial to allocate the equipment properly and would reduce the losses due to fire. Moreover, one may find the best locations

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of the barriers to block the fire across strategic paths before it surrounds the transmission lines. The problem of finding the barriers to block the fire toward some special direction is well-known as the blocking problem [4].

We study the cases in which the terrain is flat, and the fuel, temperature, humidity, etc., have been distributed smoothly across it. We consider different cases for the wind blowing across space and study each case separately. We assume that no two fire rays meet and the space across which the fire is spreading is a smooth manifold, for instance, an open subset of the Euclidean space. The first step in studying the propagation is finding the equation of a rotated ellipse [5]. We then find the angle of rotation and axes of the ellipse from the experimental data. Next, one needs to find the metric and wave rays equations from the ellipse equation and, the strategic paths. In this work, by a wave ray, we mean the path of a particle of fire, and a fire front is the perimeter of the area burnt by the fire. If the fire starts from a single point, the fire front is called a spherical fire front. Although, the results of this work are valid for the space of each dimension, due to possible applications, we concentrate on dimension two.

The rest of this work is structured as follows. Section 2 provides relevant background, including prior results that we need to establish our developments. What follows in Section 3 is the main results of this work. Section 4 addresses the application of the results by simulating the hypothetical wildfire propagation in MATLAB environment, and we provide the concluding remarks and give some suggestions for future works in Section 5.

2 Preliminaries

We recall some concepts of mathematics that we require to establish our results. For more details, see [13].

Let M be a smooth manifold of dimension 2, for instance M is an open subset of $\mathbf{R}^2$, $p = (x, y)$ a point in M, T_pM the space of all vectors tangent to M at p, and TM the collection of all vectors tangent to M; $TM = \{(p, V) : p \in M \,\text{and}\, V \in T_pM\}$. Assume that $V = (u, v) \in T_pM$ is a vector according to the canonical basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ for T_pM. A *Riemannian metric* on M is a smooth function h that to each point $p \in M$, h assigns a positive-definite inner product $h_p : T_pM \times T_pM \to \mathbb{R}$. Given a Riemannian metric h and a smooth vector field W on M that $h(W, W) < 1$, the function $F : TM \to \mathbb{R}$

$$F(V) = \frac{\sqrt{h^2(W, V) + \lambda h(V, V)}}{\lambda} - \frac{h(W, V)}{\lambda}, \tag{1}$$

where $\lambda = 1 - h(W, W)$ is called a *Randers metric* [3] and the pair (M, F) is a Randers space. The length of each piece-wise smooth curve $\gamma : [0, 1] \longrightarrow M$ on the Randers space (M, F) is $L[\gamma] := \int_0^1 F(\gamma'(t))dt$. Given two points p and q in the Randers space (M, F) , the distance from p to q is

$$d(p, q) := \inf_{\gamma} \int_0^1 F(\gamma'(t))dt, \tag{2}$$

where we take the infimum over all piecewise smooth curves $\gamma : [0, 1] \longrightarrow M$ joining p to q. A smooth curve in a Randers space is called a *geodesic* if it is locally the quickest time route joining any two nearby points on this curve. Given a Randers space (M, F), the Randers geodesics are solutions of [13]

$$\frac{d^2x^r}{dt^2} + \frac{1}{2}\sum_{l,k,j=1}^{2} g^{rl}(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^l})\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \qquad r = 1, 2, \tag{3}$$

where $[g_{rj}] = \frac{1}{2}\frac{\partial^2 F^2}{\partial v_r \partial v_j}$, $[g^{rj}]$ is the inverse matrix of $[g_{rj}]$, $(v_1, v_2) = (u, v)$, and $(x^1, x^2) = (x, y)$. Likewise, given a Riemannian space (M, h), the Riemannian geodesics are solutions of [8]

$$\frac{d^2x^r}{dt^2} + \frac{1}{2}\sum_{l,k,j=1}^{2} h^{rl}\left(\frac{\partial h_{lk}}{\partial x^j} + \frac{\partial h_{lj}}{\partial x^k} - \frac{\partial h_{kj}}{\partial x^l}\right)\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \qquad r = 1, 2, \tag{4}$$

where $[h^{rj}]$ is the inverse matrix of $[h_{rj}]$ and $(x^1, x^2) = (x, y)$. Given a vector field W, the smooth function $\varphi(t,p) : [0,1] \times M \to M$ is called the flow of W if at each point p, $\varphi(0,p) = p$ and $\frac{d\varphi}{dt}(t,p) = W(\varphi(t,p))$. The vector field $W = (W^1, W^2)$ is called *Killing* if for all $r, j = 1, 2$,

$$\sum_{k=1}^{2}\left(W^k \frac{\partial h_{rj}}{\partial x^k} + h_{kj}\frac{\partial W^k}{\partial x^r} + h_{rk}\frac{\partial W^k}{\partial x^j}\right) = 0. \tag{5}$$

For wildfire propagation, we need to calculate the ellipse equation to obtain the Riemannian/Randers metric and then the propagation model. At each point, p, the semi-axes of the ellipse are a and b, and the ellipse is rotated with the angle θ around the origin in the clockwise direction. The ellipse equation on M is [5]

$$Q((u,v),(x,y)) = \left(\frac{u\cos\theta - v\sin\theta}{a}\right)^2 + \left(\frac{u\sin\theta + v\cos\theta}{b}\right)^2 = 1, \tag{6}$$

where $\theta(x,y)$, $a(x,y)$, and $b(x,y)$ are smooth functions on M, that for the sake of clarity are written as θ, a, and b, and determined from the experimental data. From Eq. 6, we calculate the Riemannian metric by

$$h(x,y) = \frac{1}{2}\texttt{Hess}\, Q := \frac{1}{2}[Q_{v_1 v_2}], \tag{7}$$

where $(v_1, v_2) = (u, v)$ and Q_{v_i} is the partial derivative of Q with respect to v_i, $i = 1, 2$. Finally, the Randers metric associated with the propagation is calculated by Eq. (1). We recall Theorems 2.1 and 2.2 from [5] that are used to establish our result.

Remark 2.1. *It is worth mentioning that performing the calculations in Eq. (3) is much more complicated than the same job in Eq. (4). Because, in order to obtain system of Eq. (3), one first needs to find the expression of F^2- in which the square root of the Riemannian metric appears; Then, one calculates the second order partial derivatives of F^2 and obtains g_{ij}, and finally the first order partial derivatives of g_{ij} to get the system of Eq. (3). Therefore, in the literature, researchers have always tried to find a Riemannian metric associated with the Finsler metric.*

Theorem 2.1. *A wildfire spreads in M, the fuel, temperature, humidity, etc. change smoothly in M, the Killing wind W blows, and A is the fire front at time 0. Then:*

(i) The wave ray from each $p \in A$ is $\varphi(t, \alpha(t))$, where φ is the flow of W and α is the solution of Eq. (4) with initial condition $\alpha(0) = p$, $||\alpha'(t)||_h = 1$, and $d\varphi_p \alpha'(0) \underset{h}{\perp} A$. Here, $h = \frac{1}{2}\texttt{Hess}\, Q$ and

$$Q(u,v,w) = \left(\frac{u}{a}\right)^2 + \left(\frac{v\cos\theta - w\sin\theta}{b}\right)^2 + \left(\frac{v\sin\theta + w\cos\theta}{c}\right)^2 = 1, \tag{8}$$

(ii) The fire front at time τ is $\{\varphi(\tau, \alpha(\tau)) : \alpha(0) \in A\}$.

(iii) Given each area B, the strategic path toward B is $\varphi(t, \alpha(t))$ such that $\varphi(\tau, \alpha(\tau)) = q$. Here, τ is the time when the fire reaches B and q is the point of arrival.

Theorem 2.2. *A wildfire spreads in M, the fuel, temperature, humidity, etc. are distributed smoothly, the smooth vector field wind W blows, and A is the fire front at time 0. Then:*

(i) The wave rays are solutions $\gamma(t)$ of system (3) such that $\gamma'(0) \underset{F}{\perp} A$ and $F(\gamma') = 1$. Here, F is the Randers metric (1).

(ii) The fire front at time τ is $\{\gamma(\tau) : \gamma(0) \in A\}$.

(iii) Given each area B, the strategic path that meets B is the wave ray $\gamma(t)$ for which $\gamma(\tau) = q$. Here, τ is the time when the fire front intersects B for the first time and q is the point of intersection.

The following section establishes the main results to give the strategic paths.

3 Main results

Here, we assume that a wildfire spreads in a flat land M, M is an open subset of the Euclidean space, and the transmission lines of the power system are located in the region within some distance of the fire. The fuel, temperature, humidity, etc., have been distributed smoothly across the space, and a wind W, a smooth vector field, is blowing. We consider the point of initiating fire, that is the fire front at time 0, as the origin of the coordinate system. We aim to give the time when the fire surrounds the power system's transmission line and the path through which the fire reaches the line. We consider two cases for the wind: constant wind and space-dependent wind. For each case, we give the equations of strategic paths that reach the transmission line.

Theorem 3.1. *Assume that a wildfire spreads across a flat land M and the wind W blows. Then:*

1. *If W is constant satisfying*

$$\sum_{k=1}^{2} W^k \frac{\partial h_{rj}}{\partial x^k} = 0, \tag{9}$$

for all $r, j = 1, 2$, the strategic path is $\gamma(t) = tW + \alpha(t)$, where $\alpha(t)$ is the solution of system 4;

2. *If W is Killing, the strategic path is $\gamma(t) = \varphi(t, \alpha(t))$, where φ is the flow of W and $\alpha(t)$ is the solution of system 4;*

3. *If W is not Killing, the strategic path is $\gamma(t)$, where $\gamma(t)$ is the solution of system 3;*

provided that, in all of the cases, we have

i. $\gamma(0) = 0$,

ii. $||\gamma'(0) - W||_h = 1$,

iii. $\gamma(\tau) = q$.

Here, τ is the time once the perimeter of the burnt area, that is the set

$$\{\gamma(\tau) \,:\, \gamma(0) = 0, ||\gamma'(0) - W||_h = 1\}, \tag{10}$$

meets the transmission line and q is the point(s) of intersection of $\{\gamma(\tau)\}$ and the transmission line.

Proof. From the experimental data, we obtain the equation of ellipse by Eq. (6), and from the ellipse, the Riemannian and Randers metrics are obtained by Eqs. (7) and (1), respectively. According to Theorems 2.1 and 2.2, we first need to find the equation of wave rays to obtain the strategic path.

If the wind is constant and satisfies the Eq. 9, then is it a constant *Killing* vector field. Hence one applies the Theorem 2.1 to encounter the wave rays. It is not difficult to see that the flow of W is $\varphi(t,p) = tW + p$, for every point p of M. Accordingly, the wave ray initiating from p is $\gamma(t) = \varphi(t, \alpha(t)) = tW + \alpha(t)$, where α is a solution of system Eq. 4 provided that $\alpha(0) = 0$ and $||\alpha'(t)||_h = 1$. From two last equalities we have $\gamma(0) = 0$ and $||\gamma'(0) - W||_h = 1$. Next, we assume that B is the transmission line of power system, the set $\{\tau W + \alpha(\tau)\}$ is the fire front that meets B, and $q = B \cap \{\tau W + \alpha(\tau)\}$. Therefore, item (iii) of Theorem 2.1 concludes the proof of item 1.

For items 2. and 3., one follows an argument similar to above and applies Theorems 2.1 and 2.2.

□

Example 3.1. *Here, we simulate a hypothetical wildfire that spreads in woodland in Shanxi province in China and take the data and figures from [5]. We consider two different cases for the wind and ambient conditions. First, a constant wind toward the southeast blows whose speed is* $4m/s$ *for which the data are* $a = 9.8$ *and* $b = 3.98$*, and* $\theta = \pi/4$*. In the second case, the speed of wind is* $.5m/s$ *toward the east and* $a = 5/2$*,* $b = 1$*, and* $\theta = \frac{1-y}{2}$*. Figs. 1a and 1b, respectively, depict these two cases.*

For the first case, the matrix of the Riemannian metric is

$$\hbar = \begin{pmatrix} 0.0368 & 0.026 \\ 0.026 & 0.0367 \end{pmatrix}.$$

One, by applying item 1. of Theorem 3.1 and finding the set 10, finds out that the fire reaches the transmission line after 25.21 *minutes, through the segment of line* $r(t) = t(-1.64, 1.98)$ *which is shown in Fig. 1a with the red color segment that meets the line is the red star.*

For the second case, the Riemannian metric $\hbar$ *is given by (11). It is not difficult to see that the wind* $W = (.5, 0)$ *is Killing as the Riemannian metric (11) has no dependency on* x *and, therefore, all the partial derivatives in Eq. (5) become zero. In this case, the fire reaches the line after* 2 *hours through the yellow curve that reaches the red star. The wave rays are given by* $\gamma_F(t) = (.5, 0)t + (x(t), y(t))$*, where* $(x(t), y(t))$ *is the solution of the system* (4) *with the metric*

$$\hbar = \begin{pmatrix} \frac{4}{25}\cos^2(\frac{1-y}{2}) + \sin^2(\frac{1-y}{2}) & \frac{21}{48}\sin(1-y) \\ \frac{21}{48}\sin(1-y) & \frac{4}{25}\sin^2(\frac{1-y}{2}) + \cos^2(\frac{1-y}{2}) \end{pmatrix}. \tag{11}$$

The curve/line segment that reaches the red star is the strategic path through which the fire reaches the transmission lines. As one can see, in the same area but under different conditions, the shapes of propagation would be different. When the conditions (fuel distribution, temperature, humidity, etc.) are uniform across the land, and the wind is constant, the fire rays are line segments, and the propagation shape is usually a circle or ellipse. However, with even the constant wind but the nonuniform conditions, the propagation shape is not easily predictable. In this case, the mathematical techniques would help predict fire behavior more reliably.

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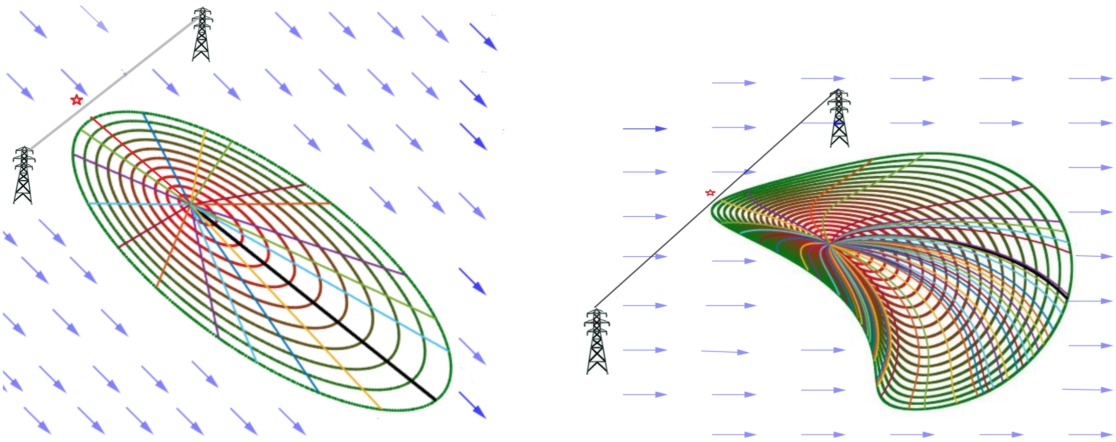

(a) Here $a = 9.8$, $b = 3.98$, $\theta = \pi/4$, and the wind is toward southeast with speed $4m/s$. The fire reaches the transmission line after 25.21 min and $\Delta t = 2.51$ min.

(b) Here $a = 5/2$, $b = 1$, $\theta = (1 - y)/2$, $W = (.5, 0)$, and $\Delta t = 12$ min. The fire reaches the transmission line after 2 hours.

Figure 1: Two propagations in the Shanxi province in the same area, however different conditions. Here, Δt is the time between two consecutive fire fronts.

4 Conclusion

In this study, we utilized Randers metric techniques to derive equations for the routes through which wildfires can reach power transmission lines. While we focused on power transmission lines due to their high applicability, our results can be applied to any area at risk of catching fire.

Identifying strategic paths can lead to more efficient fire control, ultimately reducing the damage caused by the fire. To demonstrate the potential of our approach, we provided an example of simulating two hypothetical wildfires that occur in the same area but under different transmission line conditions.

Future research could explore scenarios involving time-dependent wind patterns, non-flat terrain, and non-smooth conditions throughout the region, which are more reflective of real-world situations. Although, in the case of time-dependent cases, [2] and [9] already have some interesting results in Finsler geometry, there are still several important cases corresponding to the reality that one can investigate.

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