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Well posedness for rabies disease epidemic models for bovine and bats populations with spatial diffusion

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Abstract. In this paper, we study the well posedness to a SI epidemic models with spatial diffusion for the spreading of Rabies in the Bovine population with Bats how vector. The well-posedness of the model is proved using the Semigroup theory of sectorial operators and existence results for abstract parabolic differential equations.

keywords. Abstract Differential Equations, Rabies disease, SI Model, Semigroup Theory.

1 Introduction

In this paper, we investigated the well posedness for a SI epidemic model with spatial diffusion for transmission of Rabies by Bats in populations of Bovines, i. e., we show that solutions are continuous, globally defined and non-negative. This type of study is important in future research for qualitative and numerical studies for this system. This problem was studied in other populations of animals, see [6, 7] and references therein, but the study of transmission of Rabies in populations of Bovine with Bats how vector is a topic not treated in the literature.

In this section, we present some notations and results in functions spaces, semigroup of linear operators theory, abstract differential equations and qualitative analysis of parabolic differential equations. For more details, the reader is referred to [1, 2, 4, 5, 8, 9]. In this work, we denote by Ω a bounded domain in $\mathbb{R}^3$. For $1 \leq p \leq \infty$, the space of complex-valued L^p functions in Ω denoted by $L^p(\Omega)$ with the usual norm $\|\cdot\|_{L^p}$. The complex Sobolev space in Ω of order $k, k = 0, 1, 2, \ldots,$ is denoted by $H^k(\Omega)$ with norm $\|\cdot\|_{H^k}$. The space of complex-valued continuous functions on $\overline{\Omega}$ is denoted by $\mathcal{C}(\overline{\Omega})$ with norm $\|\cdot\|_{\mathcal{C}}$. Let X be a Banach space with norm $\|\cdot\|$, we denote by $\mathcal{C}(\Omega; X)$ and $\mathcal{C}^1(\Omega; X)$ the space of X-valued continuous functions and of X-valued continuously differentiable functions, respectively. Additionally, let $\mathcal{B}(\Omega; X)$ be the space of X-valued bounded functions. The Sobolev space of fractional order $s > 0$ is denoted by $H^s(\Omega)$ with norm $\|\cdot\|_{H^s}$. We assume Ω has a C^2 class boundary $\partial\Omega$, and for $\frac{3}{2} < s \leq 2$ by $H^s_N(\Omega)$ we denote a closed subspace of $H^s(\Omega)$ such that $H^s_N(\Omega) = \{u \in H^s(\Omega) : \partial_n u = 0 \text{ on } \partial\Omega\}$.

In what follows, for the sake of simplicity, we use the universal notation C to denote any constant that is determined for each specific occurrence of Ω. In cases in which C also depends on some parameter, say ξ, we use the notation C_ξ.

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Let us comment on the existence theorem for local solutions to an abstract equation in a Banach Space. We consider the following Cauchy problem for an abstract evolution equations in X

$$\begin{cases} \dfrac{dU}{dt} + AU &= F(U), \quad 0 < t \leq T, \\ U(0) &= U_0. \end{cases} \tag{1}$$

Here, A is a sectorial operator of X with angle $0 \leq \omega_A < \frac{\pi}{2}$. By definition,

$$\sigma(A) \subset \Sigma_\omega = \{\lambda \in \mathbb{C} :\mid arg(\lambda) \mid < \omega\}, \quad \omega_A < \omega < \frac{\pi}{2}, \tag{2}$$

and

$$\| (\lambda - A)^{-1} \| \leq \frac{M_\omega}{\mid \lambda \mid}, \quad \lambda \notin \Sigma_\omega, \quad \omega_A < \omega < \frac{\pi}{2}. \tag{3}$$

Theorem 1.1. *[8, Theorem 16.7] Let $A = -\Delta + 1$ be a second-order differential operator in $L^2(\Omega)$ with the Neumann boundary condition on $\partial\Omega$. Then, the domains of the fractional powers of A is characterized by*

$$\mathcal{D}(A^\theta) = \begin{cases} H^{2\theta}(\Omega), & \text{for } 0 \leq \theta < \frac{3}{4}, \\ H^{2\theta}_N(\Omega), & \text{for } \frac{3}{4} \leq \theta < 1, \end{cases}$$

with norm equivalence $C_\Omega^{-1} \| u \|_{H^{2\theta}(\Omega)} \leq \| A^\theta u \|_{L^2(\Omega)} \leq C_\Omega \| u \|_{H^{2\theta}(\Omega)}, \quad u \in D(A^\theta).$

The operator A defined above generates in L_2-spaces an analytic semigroup $(T(t))_{t\geq 0}$. For $\theta \geq 0$ it satisfies the estimate

$$\| A^\theta T(t) w \|_{L^2} \leq C \frac{e^{-\delta t}}{t^\theta} \| w \|_{L^2}, \quad t > 0, w \in L^2(\Omega), \tag{4}$$

with some fixed constant $\delta > 0$. For more details see [3] and the reference there in.

Let F is a nonlinear mapping from $\mathcal{D}(A^\eta)$ into X, where $0 \leq \eta < 1$. F is assumed to satisfies the Lipschitz condition of the form

$$\begin{aligned} \| F(U) - F(V) \| \quad \leq \quad & \varphi(\| U \| + \| V \|) \times [\| A^\eta (U - V) \| \\ & + (\| A^\eta U \| + \| A^\eta V \|) \| U - V \|], \quad U, V \in \mathcal{D}(A^\eta), \end{aligned} \tag{5}$$

and $\varphi(\cdot)$ is some increasing continuous function. The initial value U_0 is taken in $\mathcal{D}(A^\eta)$. For more details about local and global theorems of existence and uniqueness of solutions for abstract parabolic differential equations see [8].

2 The model

In this section, we introduce the epidemic model for the Rabies disease in Bovine and Bats populations. The total of Bovine population N_b is divided into two sub-populations: Bovines that may become infected (susceptible S_b); Bovines infected by rabies (infected I_b);

The parameter η is the birth rate of Bovine. The birth rate is assumed to be equal to natural death. The total population of Bats N_m is divided into two sub-populations: bats which may become infected by the disease; bats infected by the Rabies.

The parameter μ is the birth rate of the bats and it is assumed to be equal to the death rate. A Bovine infected I_b can transmit to the susceptible Bats S_m because of an effective transmission with a rate κ. A bat infected I_m can transmit to the a susceptible bovine S_b with a transmission rate β.

A susceptible bat can be infected if there exists an contact with an infected bats with a rate λ. d represents the diffusion coefficient of bats. The rate τ represents the additional mortality in bovine population caused by Rabies transmitted by Bats. Homogeneous mixing is assumed; that is, all susceptible Bovines have the same probability to be infected and all susceptible Bats have the same probability to be infected.

We can describe the disease epidemic models with spatial diffusion given by the following partial differential equations of parabolic type with Neumann condition:

$$\left\{\begin{array}{rcl} S_m' &=& \mu N_m + d\Delta S_m - \kappa S_m I_b - \lambda S_m I_m - \mu S_m, \text{ in } (0,\infty)\times\Omega, \\ I_m' &=& d\Delta I_m + \kappa S_m I_b + \lambda S_m I_m - \mu I_m, \text{ in } (0,\infty)\times\Omega, \\ S_b' &=& \eta N_b - \beta S_b I_m - \eta S_b, \text{ in } (0,\infty)\times\Omega, \\ I_b' &=& \beta S_b I_m - (\eta+\tau) I_b, \text{ in } (0,\infty)\times\Omega, \end{array}\right. \tag{6}$$

$$S_m(0,x) = S_{m,0}(x), \quad I_m(0,x) = I_{m,0}(x), S_b(0,x) = S_{b,0}(x), \quad I_b(0,x) = I_{b,0}(x), \quad x\in\Omega, \tag{7}$$

Let $\mathcal{U} = \begin{pmatrix} S_m \\ I_m \\ S_b \\ I_b \end{pmatrix}$, we get the problem (6)-(7) can be formulated as abstract Cauchy problem

$$\mathcal{U}'(t) + \mathcal{A}\mathcal{U}(t) = \mathcal{F}(\mathcal{U}), \ t>0, \tag{8}$$

$$\mathcal{U}(0) = \mathcal{U}_0 \in X, \tag{9}$$

where

$$\mathcal{A} = \begin{pmatrix} -d\Delta+\mu & 0 & 0 & 0 \\ 0 & -d\Delta+\mu & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta+\tau \end{pmatrix} \text{ and } \mathcal{F}(\mathcal{U}) = \begin{pmatrix} \mu N_m - \kappa S_m I_b - \lambda S_m I_m \\ \kappa S_m I_b + \lambda S_m I_m \\ \eta N_b - \beta S_b I_m \\ \beta S_b I_m \end{pmatrix}.$$

In (8)-(9), the space X is defined by $X = L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)$ under the norm

$$\left\| \begin{pmatrix} S_m \\ I_m \\ S_b \\ I_b \end{pmatrix} \right\| = \left(\int_\Omega |S_m|^2 + |I_m|^2 + |S_b|^2 + |I_b|^2 \, dx \right)^{\frac{1}{2}},$$

and we define the $D(\mathcal{A})$ by $D(\mathcal{A}) = \left\{ \begin{pmatrix} S_m \\ I_m \\ S_b \\ I_b \end{pmatrix} \in X : S_m, I_m \in H_N^2(\Omega) \text{ and } S_b, I_b \in L^2(\Omega) \right\}$.

2.1 Global existence, positivity of solutions

In the sequel, we show the existence of local solutions associated to the system (6)-(7).

Theorem 2.1. *For each initial function data $(S_{m,0}, I_{m,0}, S_{b,0}, I_{b,0}) \in L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)$, with $S_{m,0}, I_{m,0}, S_{b,0}, I_{b,0} \geq 0$. Then, the problem (8)-(9) admits a unique local-in-time solution $\mathcal{U} = (S_m, I_m, S_b, I_b)$ in the space $\mathcal{U} \in \mathcal{C}((0,T_{\mathcal{U}_0}]; D(\mathcal{A})) \cap \mathcal{C}^1((0,T_{\mathcal{U}_0}]; X) \cap \mathcal{C}([0,T_{\mathcal{U}_0}]; X)$, where $T_{\mathcal{U}_0}$ is a positive constant depending only of $\| \mathcal{U}_0 \|$.*

Proof. Let A_S and A_I are realizations of operators $-d\Delta+\mu$ and $-d\Delta+\mu$ respectively, in $L^2(\Omega)$ under the Neumann boundary conditions on $\partial\Omega$. It is not difficult to show $\mathcal{A}$ is a positive defined self-adjoint operators of X. For $0<\theta<1$, its fractional power $\mathcal{A}^\eta$ are also diagonal operator $\mathcal{A}^\theta=\begin{pmatrix} A_S^\theta & 0 & 0 & 0\\ 0 & A_I^\theta & 0 & 0\\ 0 & 0 & \eta^\theta & 0\\ 0 & 0 & 0 & (\eta+\tau)^\theta\end{pmatrix}$. By [8, Theorem 16.7] and [8, Theorem 16.9], their domains are characterized by

$$D(\mathcal{A}^\theta)=\{{}^t(S_m,I_m,S_b,I_b): S_m,I_m\in H^{2\theta}(\Omega) \text{ and } S_b,I_b\in L^2(\Omega)\},\quad 0\le\theta\le\frac{3}{4},$$

and $D(\mathcal{A}^\theta)=\{{}^t(S_m,I_m,S_b,I_b): S_m,I_m\in H_N^{2\theta}(\Omega) \text{ and } S_b,I_b\in L^2(\Omega)\}$, $\frac{3}{4}<\theta\le 1$.

Let

$$\mathcal{F}(\mathcal{U}_i)=\begin{pmatrix}\mu N_m-\kappa S_m^iI_b^i-\lambda S_m^iI_m^i\\ \kappa S_m^iI_b^i+\lambda S_m^iI_m^i\\ \eta N_b-\beta S_b^iI_m^i\\ \beta S_b^iI_m^i\end{pmatrix},$$

with domain $D(\mathcal{F})=\{{}^t(S_m,I_m,S_b,I_b): S_m,I_m\in L^\infty(\Omega) \text{ and } S_b,I_b\in L^2(\Omega)\}$ where ${}^tM$ represent the transpose of M.

Fix an exponent such that $\frac{3}{4}<\theta<1$. Then, by [8, Theorem 1.36] we have $D(\mathcal{A}^\theta)\subset D(\mathcal{F})$. Therefore, for $\mathcal{U}_1,\mathcal{U}_2\in D(\mathcal{F})$.

$$\begin{aligned}\|\mathcal{F}(\mathcal{U}_1)-\mathcal{F}(\mathcal{U}_2)\|^2 &\le C(\|S_m^1\|_{L^\infty}^2\|I_b^1-I_b^2\|_{L^2}^2+\|S_m^1-S_m^2\|_{L^\infty}^2\|I_b^2\|_{L^2}^2\\ &\quad+\|S_m^1\|_{L^\infty}^2\|I_m^1-I_m^2\|_{L^2}^2+\|S_m^1-S_m^2\|_{L^2}^2\|I_m^2\|_{L^\infty}^2\\ &\quad+\|S_b^1\|_{L^2}^2\|I_m^1-I_m^2\|_{L^\infty}^2+\|S_b^1-S_b^2\|_{L^2}^2\|I_m^2\|_{L^\infty}^2)\\ &\le C((\|A^\theta S_m^1\|_{L^2}^2+\|A^\theta I_m^2\|_{L^2}^2)\times\\ &\quad(\|S_m^1-S_m^2\|_{L^2}^2+\|S_b^1-S_b^2\|_{L^2}^2+\|I_m^1-I_m^2\|_{L^2}^2+\|I_b^1-I_b^2\|_{L^2}^2)\\ &\quad+(\|S_b^1\|_{L^2}+\|I_b^2\|_{L^2})^2(\|A^\theta(S_m^1-S_m^2)\|_{L^2}^2+\|A^\theta(I_m^1-I_m^2)\|_{L^2}^2))\\ &\le C(\|A^\theta\mathcal{U}_1\|+\|A^\theta\mathcal{U}_2\|)^2\|\mathcal{U}_1-\mathcal{U}_2\|^2\\ &\quad+(\|\mathcal{U}_1\|+\|\mathcal{U}_2\|)^2\|A^\theta(\mathcal{U}_1-\mathcal{U}_2)\|^2).\end{aligned}$$

By [8, Theorem 4.4], the problem (6)-(7) has a unique local solution in the function space $\mathcal{U}\in\mathcal{C}((0,T_{\mathcal{U}_0}];D(\mathcal{A}))\cap\mathcal{C}^1((0,T_{\mathcal{U}_0}];X)\cap\mathcal{C}([0,T_{\mathcal{U}_0}];X)$. □

Theorem 2.2. *For any given initial data satisfying the condition (6)-(7), there exists a unique solution of problem defined on* $t\in[0,T_{\mathcal{U}_0}]$ *and this solution remains nonnegative for all* $t\in[0,T_{\mathcal{U}_0}]$.

Proof. We will show that $S_m(t)\ge0, I_m(t)\ge0, S_b(t)\ge0$ and $I_b(t)\ge0$ for all $0<t\le T_{\mathcal{U}_0}$. For this purpose, however, we have to introduce the modified nonlinear operator

$$\mathcal{F}(\tilde{\mathcal{U}})=\begin{pmatrix}\mu N_m-\kappa\tilde{S}_m\chi(Re\tilde{I}_b)-\lambda\tilde{S}_m\tilde{I}_m\\ \kappa\tilde{S}_m\chi(Re\tilde{I}_b)+\lambda\tilde{S}_m\tilde{I}_m\\ \eta N_b-\beta\tilde{S}_b\tilde{I}_m\\ \beta\tilde{S}_b\tilde{I}_m\end{pmatrix},$$

where $\chi(u)$ denotes a function such that $\chi(u)\equiv0$ for $-\infty<u<0$ and $\chi(u)=u$ for $0\le u<\infty$. We have to consider the auxiliar problem

$$\tilde{\mathcal{U}}'(t)+\mathcal{A}\tilde{\mathcal{U}}(t)=\mathcal{F}(\tilde{\mathcal{U}}),\ t>0, \tag{10}$$

$$\tilde{\mathcal{U}}(0)=\mathcal{U}_0\in X, \tag{11}$$

It is clear that the new nonliner operator $\tilde{\mathcal{F}}$ also satisfies (5) with the same expoent θ because $\| \chi(Re\ u) - \chi(Re\ v) \|_{L^2} \leq \| u - v \|$ for $u, v \in L^2(\Omega)$. Therefore (10)-(11) possesses a unique local solution $\tilde{\mathcal{U}} = (\tilde{S}_m, \tilde{I}_m, \tilde{S}_b, \tilde{I}_b)$ on an interval $[0, \tilde{T}_{\mathcal{U}_0}]$ in the same functions spaces $\tilde{S}_m, \tilde{I}_m \in \mathcal{C}([0, \tilde{T}_{\mathcal{U}_0}]; L^2(\Omega)) \cap \mathcal{C}^1((0, \tilde{T}_{\mathcal{U}_0}]; L^2(\Omega)) \cap \mathcal{C}((0, \tilde{T}_{\mathcal{U}_0}]; H^2_N(\Omega))$ and $\tilde{S}_b, \tilde{I}_b \in \mathcal{C}([0, \tilde{T}_{\mathcal{U}_0}]; L^2(\Omega)) \cap \mathcal{C}^1((0, \tilde{T}_{\mathcal{U}_0}]; L^2(\Omega))$. First, we will show that $\tilde{S}_m(t) \geq 0, \tilde{I}_m(t) \geq 0, \tilde{S}_b(t) \geq 0$ and $\tilde{I}_b(t) \geq 0$ for all $0 < t \leq \tilde{T}_{\mathcal{U}_0}$.

We note that $\tilde{\mathcal{U}}(t)$ is real-valued. Indeed, the complex conjugate $\overline{\tilde{\mathcal{U}}(t)}$ of $\tilde{\mathcal{U}}(t)$ is also a local solution of (10)-(11) with the same initial value $\mathcal{U}_0$. From the uniqueness of solutions, $\overline{\tilde{\mathcal{U}}(t)} = \tilde{\mathcal{U}}(t)$, hence $\tilde{\mathcal{U}}(t)$ is real-valued.

Let $H(\cdot)$ be a $C^{1,1}$ cutoff function such that $H(u) = \frac{u^2}{2}$ for $-\infty < u < 0$ and $H(u) \equiv 0$ for $0 \leq u < \infty$. By Yagi [8] (page 52), the function $\psi(t) = \int_\Omega H(u(t))dx$ is continuously differentiable.

Computing the derivative of $\psi(t)$ with $u(t) = \tilde{S}_m(t)$, we get

$$\psi'(t) = \int_\Omega H'(\tilde{S}_m)\tilde{S}'_m dx = \int_\Omega H'(\tilde{S}_m)\left(\mu N_m + d\Delta\tilde{S}_m - \kappa\tilde{S}_m\chi(\tilde{I}_b) - \lambda\tilde{S}_m\tilde{I}_m - \mu\tilde{S}_m\right)dx.$$

But, (by property (1.96) of [8]) we can get

$$\begin{aligned}\int_\Omega H'(\tilde{S}_m)\Delta\tilde{S}_m dx &= -\int_\Omega \nabla H'(\tilde{S}_m)\cdot\nabla\tilde{S}_m dx \\ &= -\int_\Omega \nabla H'(\tilde{S}_m)\cdot\nabla H'(\tilde{S}_m)dx = -\int_\Omega |\nabla H'(\tilde{S}_m)|^2\,dx \leq 0,\end{aligned}$$

therefore

$$\begin{aligned}\psi'(t) &= -d\int_\Omega |\nabla H'(\tilde{S}_m)|^2\,dx + \mu N_m\int_\Omega H'(\tilde{S}_m)dx - \kappa\int_\Omega H'(\tilde{S}_m)\tilde{S}_m\chi(\tilde{I}_b)dx \\ &\quad -\int_\Omega H'(\tilde{S}_m)\tilde{S}_m(\lambda\tilde{I}_m + \mu)dx.\end{aligned}$$

Since $H'(\tilde{S}) \leq 0$,

$$\begin{aligned}\psi'(t) &\leq -\int_\Omega H'(\tilde{S}_m)\tilde{S}_m(\lambda\tilde{I}_m + \mu)dx \leq C \| H'(\tilde{S}_m)\tilde{S}_m \|_{L^1} (1 + \| \tilde{I}_m \|_{L^\infty}) \\ &\leq C \| H(\tilde{S}_m) \|_{L^1} (1 + \| \tilde{S}_m \|_{L^\infty} + \| \tilde{I}_m \|_{L^\infty} + \| \tilde{I}_b \|_{L^2} + \| \tilde{I}_b \|_{L^\infty}) \\ &\leq C \| H(\tilde{S}_m) \|_{L^1} (1 + \| \tilde{S}_m \|_{H^{2\theta}} + \| \tilde{I}_m \|_{H^{2\theta}} + \| \tilde{S}_b \|_{L^2} + \| \tilde{I}_b \|_{L^2}) \\ &= C\psi(t)(1 + \| \tilde{S}_m \|_{H^{2\theta}} + \| \tilde{I}_m \|_{H^{2\theta}} + \| \tilde{S}_b \|_{L^2} + \| \tilde{I}_b \|_{L^2}).\end{aligned}$$

Therefore $\psi'(t) \leq C\psi(t)(1 + \| \mathcal{A}^\eta\tilde{\mathcal{U}}(t) \|)$. Thus, by Lemma Gronwall,

$$\psi(t) \leq \psi(0)e^{C\int_0^t (1 + \|\mathcal{A}^\theta\tilde{\mathcal{U}}(\tau)\|)d\tau}.$$

Using the bound $\| \mathcal{A}^\theta\tilde{\mathcal{U}}(\tau) \| \leq C_{\mathcal{U}_0}\tau^{-\theta}$, which means that $\| \mathcal{A}^\theta\tilde{\mathcal{U}}(\tau) \|$ is integrable in $0 \leq t \leq \tilde{T}_{\mathcal{U}_0}$. Hence, $\psi(0) = 0$ implies $\psi(t) \equiv 0$, namely $\tilde{S}_m(t) \geq 0$ for $0 \leq t \leq \tilde{T}_{\mathcal{U}_0}$.

Now, computing the derivative of $\psi(t)$ with $u(t) = \tilde{I}_m(t)$

$$\begin{aligned}\psi'(t) &= \int_\Omega H'(\tilde{I}_m)\left(d\Delta\tilde{I}_m + \kappa\tilde{S}_m\chi(\tilde{I}_b) + \lambda\tilde{S}_m\tilde{I}_m - \mu\tilde{I}_m\right)dx = -d\int_\Omega |\nabla H'(\tilde{I}_m)|^2\,dx \\ &+\kappa\int_\Omega H'(\tilde{I}_m)\tilde{S}_m\chi(\tilde{I}_b)dx - 2\lambda\int_\Omega H(\tilde{I}_m)\tilde{S}_m dx - 2\mu\int_\Omega H(\tilde{I}_m)dx \leq 0.\end{aligned}$$

By $H(\tilde{I}_m) \geq 0$ and $H'(\tilde{I}_m) < 0$ we get $\psi'(t) \leq 0$. From $\psi(0) = 0$, we get $\psi(t) \equiv 0$, this implies $\tilde{I}_m(t) \geq 0$ for $0 \leq t \leq \tilde{T}_{\mathcal{U}_0}$. For $\psi(t)$ with $u(t) = \tilde{S}_b(t)$ we have $\psi'(t) = \int_\Omega H'(\tilde{S}_b)\tilde{S}'_b(t)dx = \int_\Omega \tilde{H}'(S_b)\left(\eta N_b - \beta\tilde{S}_b\tilde{I}_m - \eta\tilde{S}_b\right)dx \leq 0$.

From the same argument before, we get $\tilde{S}_b(t) \geq 0$ for $0 \leq t \leq \tilde{T}_{\mathcal{U}_0}$. For $\psi(t)$ with $u(t) = \tilde{I}_b(t)$ we have $\psi'(t) = \int_\Omega H'(\tilde{I}_b)\tilde{I}'_b(t)dx = \int_\Omega \tilde{H}'(I_b)\left(\beta\tilde{S}_b\tilde{I}_m - (\eta+\tau)\tilde{I}_b\right)dx \leq 0$. therefore $\tilde{I}_b(t) \geq 0$ for $0 \leq t \leq \tilde{T}_{\mathcal{U}_0}$.

We now notice $\chi(\tilde{I}_b(t)) = \tilde{I}_b(t)$, this implies $\tilde{\mathcal{U}}$ is a local solution of the original problem (8)-(9) too. The uniqueness of solution then implies $\tilde{\mathcal{U}} = \mathcal{U}$. Hence $S_m(t) \geq 0, I_m(t) \geq 0, S_b(t) \geq 0$ and $I_b(t) \geq 0$ for $0 < t \leq \tilde{T}_{\mathcal{U}_0}$. Now we have the possibilities: If $\tilde{T}_{\mathcal{U}_0} \geq T_{\mathcal{U}_0}$ we finished the proof. If not, we define $T_0 = \sup\{0 < T \leq T_{\mathcal{U}_0} : S_m(t) \geq 0, I_m(t) \geq 0, S_b(t) \geq 0 \text{ and } I_b(t) \geq 0 \text{ for every } 0 < t \leq T\}$. From

$$\int_\Omega H(S_m(T_0))dx = \lim_{t\to T_0^-}\int_\Omega H(S_m(t))dx = 0,$$

we see that $S_m(T_0) \geq 0$. By similar argument, we have $I_m(T_0) \geq 0, S_b(T_0) \geq 0$ and $I_b(T_0) \geq 0$. So if, $T_0 = T_{\mathcal{U}_0}$, we finished the proof. If $T_0 < T_{\mathcal{U}_0}$, we will consider again the problem (10) but with the initial time T_0 and the initial value $\mathcal{U}(T_0)$. Repeating the same argument as above, we conclude that there is $\delta > 0$ such that $S_m(t) \geq 0, I_m(t) \geq 0, S_b(t) \geq 0$ and $I_b(t) \geq 0$ for every $T_0 \leq t \leq T_0 + \delta$. This is a contradiction, hence $T_0 = T_{\mathcal{U}_0}$. The above arguments implies the nonnegativity of solutions for all $0 \leq t \leq T_{\mathcal{U}_0}$. □

2.2 Boundedness of solutions

Now we show the existence of global solutions for the problem (6)-(7).

Theorem 2.3. *For any given initial data satisfying the condition (6)-(7), there exists a unique solution of problem defined on $[0,\infty)$ and this solution remains nonnegative and bounded for all $t \geq 0$.*

Proof. How S_m, I_m, S_b and $I_b \in \mathcal{C}([0, T_{\mathcal{U}_0}]; L^2(\Omega))$ we have S_m, I_m, S_b and I_b is bounded in L^2 norm in $[0,T]$ with $T < T_{\mathcal{U}_0}$. For all $t \geq T$, we have

$$\begin{aligned}
\int_\Omega S_m S'_m dx &= \int_\Omega dS_m\Delta S_m dx + \int_\Omega \mu N_m S_m - \mu S_m^2 - \kappa S_m^2 I_b - \lambda S_m^2 I_m dx,\\
\int_\Omega I_m S'_m dx &= \int_\Omega dI_m\Delta S_m dx + \int_\Omega \mu N_m I_m - \mu I_m S_m - \kappa I_m S_m I_b - \lambda S_m I_m^2 dx,\\
\int_\Omega I_m I'_m dx &= \int_\Omega dI_m\Delta I_m dx + \int_\Omega \kappa S_m I_m I_b + \lambda S_m I_m^2 - \mu I_m^2 dx,\\
\int_\Omega S_m I'_m dx &= \int_\Omega dS_m\Delta I_m dx + \int_\Omega \kappa S_m^2 I_b + \lambda S_m^2 I_m - \mu S_m I_m dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (S_m+I_m)^2 dx = -\int_\Omega d \mid \nabla S_m + \nabla I_m \mid^2 dx + \int_\Omega \mu N_m(S_m+I_m) - \mu(S_m+I_m)^2 dx\\
\leq\ & -\int_\Omega\left(\sqrt{\frac{\mu}{2}}(S_m+I_m) - \frac{1}{2}\sqrt{\frac{2}{\mu}}\mu N_m\right)^2 dx - \frac{\mu}{2}\int_\Omega \mid S_m+I_m \mid^2 dx + \int_\Omega \frac{\mu N_m^2}{2}dx\\
\leq\ & -\frac{\mu}{2}\int_\Omega \mid S_m+I_m \mid^2 dx + \frac{\mu N_m^2}{2}\mid \Omega \mid.
\end{aligned}$$

By Lemma Gronwall, $\| S_m(t)+I_m(t) \|_{L^2}^2 \leq \| S_m(0)+I_m(0) \|_{L^2}^2 \ e^{-\mu t} + \frac{N_m^2}{\mu} \mid \Omega \mid$. From $S_m(t)\geq 0$ and $I_m(t)\geq 0$, follows that $\| S_m(t) \|_{L^2}^2 + \| I_m(t) \|_{L^2}^2 \leq \| S_m(0)+I_m(0) \|_{L^2}^2 \ e^{-\mu t} + \frac{N_m^2}{\mu} \mid \Omega \mid$.

$$\begin{aligned} \int_\Omega S_b S_b' dx &= \int_\Omega \eta N_b S_b - \beta S_b^2 I_m - \eta S_b^2 dx, \quad \int_\Omega I_b S_b' dx = \int_\Omega \eta N_b I_b - \beta I_b S_b I_m - \eta I_b S_b dx, \\ \int_\Omega I_b I_b' dx &= \int_\Omega \beta I_b S_b I_m - (\eta+\tau) I_b^2 dx, \quad \int_\Omega S_b I_b' dx = \int_\Omega \beta S_b^2 I_m - (\eta+\tau) S_b I_b dx. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (S_b+I_b)^2 dx &\leq \int_\Omega \eta N_b (S_b+I_b) - \eta (S_b+I_b)^2 dx \\ &\leq -\int_\Omega \left(\sqrt{\frac{\eta}{2}}(S_b+I_b) - \frac{1}{2}\sqrt{\frac{2}{\eta}}\eta N_b \right)^2 dx - \frac{\eta}{2}\int_\Omega \mid S_b+I_b \mid^2 dx + \int_\Omega \frac{\eta N_b^2}{2} dx \\ &\leq -\frac{\eta}{2}\int_\Omega \mid S_b+I_b \mid^2 dx + \frac{\eta N_b^2}{2} \mid \Omega \mid . \end{aligned}$$

By Lemma Gronwall, $\| S_b(t)+I_b(t) \|_{L^2}^2 \leq \| S_b(0)+I_b(0) \|_{L^2}^2 \ e^{-\eta t} + \frac{N_b^2}{\eta} \mid \Omega \mid$. From $S_b(t)\geq 0$ and $I_b(t)\geq 0$, we get that $\| S_b(t) \|_{L^2}^2 + \| I_b(t) \|_{L^2}^2 \leq \| S_b(0)+I_b(0) \|_{L^2}^2 \ e^{-\eta t} + \frac{N_b^2}{\eta} \mid \Omega \mid$. The result now is consequence of the [8, Corollary 4.3]. □

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