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Caputo Derivative as Weighted Average of Historical Values: some consequences illustrated via COVID-19 data

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Abstract. This paper uses the formula previously proposed by the authors themselves, in which the Caputo fractional derivative is written proportionally to the weighted average of historical values of the classical derivative (Equation (2)). Three consequences of this formula are treated in this work. The first explicitly shows the dimension of the Caputo derivative, the second indicates which historical values of the classical derivative have greater/lower weight for the Caputo operator at the current instant, and finally, the third shows that the Caputo derivative is zero at instants after the critical point occurred (allowing interpretations for the order of the derivative, for example in the dynamics of some disease). To illustrate these three results, we used examples previously obtained by the authors themselves, modeling the curve of active COVID-19 cases with the SIR model. This approach captures the memory effect well in epidemiological models.

Key-words. Caputo Derivative, Weighted Average, Dimensional Analysis, critical state, COVID.

1 Introduction

Fractional calculus is an interesting area, primarily because it generalizes classical calculus. Furthermore, it is possible to show that operators such as Riemann-Liouville integral and derivative and Caputo derivative have a memory effect in their composition. The Caputo derivative is an general choice for mathematical modeling since it has the property that the derivative of a constant is zero, which generally does not occur with other fractional operators. Another advantage of the Caputo derivative is that, in initial value problems with Caputo fractional differential equations, the initial data can be the same as in its version with ordinary differential equations [2, 9].

As memory is present in many phenomena, especially in biomathematics, fractional calculus provides a interesting description of actual data. Furthermore, a system with fractional differential equations has one more parameter that can even be fitted: the order of the derivative. Thus, we have several possible solutions, increasing the chance of finding the ideal one to model the desired phenomenon. Furthermore, the effect of neglected parameters in usual modeling can be incorporated into the order of the derivative [2, 4, 6, 9].

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In the fractional calculus literature, there are some approaches to verify the memory effect in operators or fractional systems, as we see for example in [7, 8]. In this paper, we remember a formula presented for the first time by the authors themselves in [1] that shows more explicitly the memory effect in Caputo fractional derivative (Equation (2)). Through it, the Caputo derivative is written as proportional to the mean (through statistical expectation), weighted by the Beta distribution, of historical values of the classical derivative.

The main objective of this work is, after remembering the equation that indicates the memory effect in the Caputo derivative published by the authors in [1, 5], to show three interesting results that we can obtain from this formula: the explicit elaboration of the dimensional analysis of the fractional derivative, the possibility of observing how the weighted average works and, finally, the fact that the Caputo derivative is not zero at the critical point of the function. We do this in Section 2 and we illustrate the results in Section 3, using an application also already made by the authors in [1, 5], with the SIR epidemiological model to study active cases of COVID-19.

2 Results on Caputo Fractional Derivative

From the definitions of the Riemann-Liouville fractional derivative and integral [7], we can define the Caputo derivative as follows, for $f \in AC_{loc}([a,\infty),\mathbb{R})$:

$$(cD^{\alpha}_{a^+}f)(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha} f'(s)ds, \;\; \forall t \in (a,\infty). \tag{1}$$

Previously, the authors demonstrated that Caputo fractional derivative can be written as in the Equation (2), that is, proportional to a weighted average of the past values of the classical derivative of the operated function:

$$cD^{\alpha}_t f(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} E\left[f'(tW)\right], \;\text{ if }\; 0<\alpha<1, \tag{2}$$

where W is a random variable with the beta distribution $W \sim B(1, 1-\alpha)$.

The proof follows by the change of variable $w = \frac{s}{t}$ [1, 5]:

$$\begin{aligned} cD^{\alpha}_t f(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} f'(s)ds = \frac{1}{\Gamma(1-\alpha)}\int_0^t t^{-\alpha}\left(1-\frac{s}{t}\right)^{-\alpha} f'(s)ds \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^1 t^{1-\alpha}\,(1-w)^{-\alpha}\, f'(tw)dw \\ &= \frac{t^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\int_0^1 \frac{(1-w)^{(1-\alpha)-1}}{B(1,1-\alpha)} f'(tw)dw = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}E[f'(tW)]. \end{aligned} \tag{3}$$

In [1, 5] it is possible to see formulas that relate Riemann-Liouville fractional derivative and integral operators with weighted averages of historical values as well.

The Formula (2) highlights the memory effect on this fractional operator. Furthermore, this formula has interesting consequences, some of which we show in the following subsections.

2.1 Dimensional Analysis of the Caputo Derivative

A first useful consequence that follows directly from Equation (2) is the dimensional analysis of the Caputo derivative. In [2] it is stated that it is necessary to make changes in the differential equation with the Caputo derivative (for example, raising parameters to α) to obtain a correct dimensional analysis since this derivative has dimension $(\text{time})^{-\alpha}$ (that is, $T^{-\alpha}$). In this subsection, we show this explicitly through the Formula (2). An example is described in Subsection 3.1.

If f represents population (with dimension P), let us show that $cD_t^\alpha f(t)$ has dimension $\frac{P}{T^\alpha}$. Note that the Equation (2) can be rewritten as follows:

$$cD_t^\alpha f(t) = \Gamma(2-\alpha)\left[\frac{E\left[f'(tW)\right]}{t^{\alpha-1}}\right]. \tag{4}$$

We know that in calculating the average of values, they must have the same dimension. Thus, the average of values of $f'(t)$ (given by $E\left[f'(tW)\right]$) has dimension $\frac{P}{T}$, as does $f'(t)$. Since $\frac{1}{t^{\alpha-1}}$ has dimension $\frac{1}{T^{\alpha-1}}$, then $cD_t^\alpha f(t)$ has dimension $\frac{P/T}{T^{\alpha-1}} = \frac{P}{T^\alpha}$.

The same reasoning follows for the case of f having another dimension or being dimensionless.

2.2 Influence of Beta Distribution on Fractional Operators of Order α

Since each historical value of $f'(t)$ (before t) acts differently to define the Caputo derivative operator at the current instant t, according to the distribution $B(1, 1-\alpha)$, we investigate which of them most contribute to these operators in relation to the order $\alpha \in (0, 1)$.

From the Formula (2), it can be seen that values close to t contribute more to defining the current instant than remote values (those evaluated for times close to $t = 0$). This is justified by the fact that the probability density function $f_W(w) = \dfrac{(1-w)^{-\alpha}}{B(1, 1-\alpha)}$ is increasing when $0 < \alpha < 1$.

In Figure 1 we see that in any scenario the recent values (*i.e.*, when $w \simeq 1$) have greater weight than the historical values (*i.e.*, when $w \simeq 0$). However, the difference between such weights increases the closer the value of α is to 1, and decreases the smaller the value of α is.

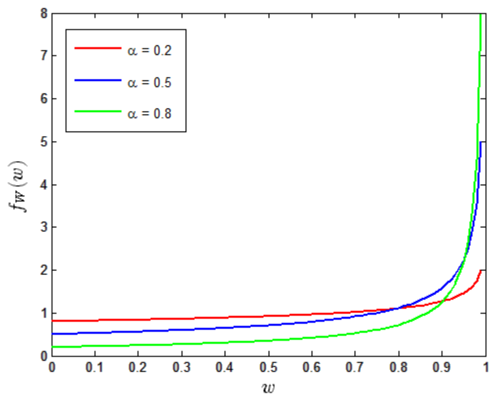

Figure 1: The density function f_W with historical weight distribution for the Caputo derivative.

This allows us to conclude that when $\alpha \sim 1$ the memory effect is small, since historical values become less important compared to recent values. On the other hand, if $\alpha \sim 0$ the historical values become more significant, characterizing a greater memory effect.

2.3 When Caputo Derivative is Zero

In this section we show another interesting interpretation that we can make directly of the Formula (2): the Caputo derivative is zero at an instant t_α (for each value of $\alpha \in (0, 1)$) after the instant t^* where the critical point occurs. That is, through the fractional derivative it is not possible to analyze the equilibrium of the function in the classical way and, moreover, the derivative is zero when the critical point has already occurred.

In classical calculus it is common to analyze the growth of a function through its derivative. However, through Formula (2) it is possible to show that the Caputo fractional derivative is not always positive when the curve is increasing and negative when the curve is decreasing. Consequently, when this derivative is null, we do not necessarily have inflection point of the function.

Let us assume that f has only one local critical point at $t^* \in [0, b]$. If $cD_t^\alpha f(t_\alpha) = 0$, for some $t_\alpha \in [0, b]$, then for (2) we have $E[f'(t_\alpha W)] = 0$. Hence, there are $t_1, t_2 \in [0, t_\alpha]$ such that $f'(t_1) > 0$ and $f'(t_2) < 0$. Thereby, $t_1 < t^* < t_2$, if the critical point is a local maximum point, or $t_2 < t^* < t_1$, if it is a local minimum point. Therefore, $t^* \in (0, t_\alpha)$, that is, at t_α the critical point has already occurred.

Example 2.1. Let $f(t) = t - t^2$. Then, $f'(t) = 1 - 2t$. This function has a single critical point (local maximum point) at $t^* = \frac{1}{2}$. To determine $cD_t^\alpha f(t)$, we use Equation (2). Due to the linearity of the operator cD_t^α, we have

$$cD_t^\alpha f(t) = cD_t^\alpha (t - t^2) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \left[1 - \frac{2t}{2-\alpha}\right]. \tag{5}$$

Thus, $cD_t^\alpha f(t_\alpha) = 0 \Leftrightarrow t_\alpha = 0$ or $t_\alpha = 1 - \frac{\alpha}{2}$. First, from (2) it is worth noting that in $t = 0$ we always have $cD^\alpha f(t) = 0$, not indicating an inflection point. Excluding this case, note that when $\alpha = 1$ we have $t_\alpha = 1 - \frac{1}{2} = t^*$ and, if $\alpha < 1$, then $t_\alpha > t^*$. That is, the time t_α in which the fractional derivative is null occurs after the time t^* in which the classical derivative is null, which is where the local maximum point occurs.

This is best illustrated in Figure 2, where we see the Caputo derivative curves $cD_t^\alpha f(t)$ for different values of α, including the case $\alpha = 1$. It is possible to see that in fact the classical derivative reaches zero at an earlier time, and the smaller the value of α is, the longer the time for the fractional derivative to be null. We see that, in fact, in $t^* = 0.5$ only the classical derivative ($\alpha = 1$) is null. Also, corroborating the results presented above, we see that the fractional derivative is positive for values of t such that $f(t)$ is decreasing.

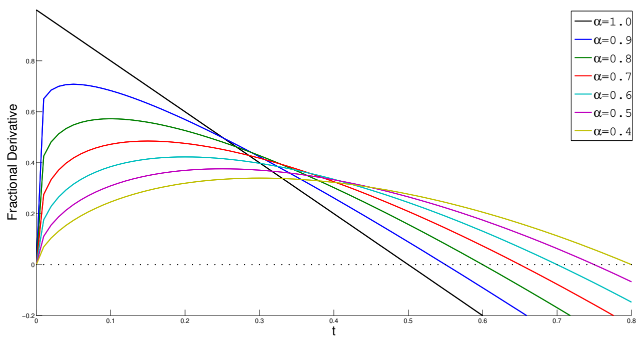

Figure 2: Fractional derivative curves $cD_t^\alpha (t - t^2)$ for different orders α.

The fact that Caputo derivative is not necessarily zero at critical points (local maximum or minimum points) means that equilibria of fractional differential equations do not necessarily coincide with points where Caputo derivative is equal to zero.

In the following section, we use the Caputo derivative to include memory effect in the SIR epidemiological model, studying data from active cases of COVID-19. With this, we illustrate the concepts presented in this section.

3 Fractional SIR Model: an application to COVID-19

In the works [1, 5] the authors used Caputo fractional derivative to include the memory effect in the SIR (Susceptible-Infected-Recovered) epidemiological model, obtaining the following system.

$$\left\{ \begin{array}{rcl} cD_t^\alpha S(t) & = & -\beta^\alpha S(t)I(t) \\ cD_t^\alpha I(t) & = & \beta^\alpha S(t)I(t) - \gamma^\alpha I(t) \\ cD_t^\alpha R(t) & = & \gamma^\alpha I(t) \end{array} \right. , \tag{6}$$

where β is the transmission rate, γ is the recovery rate [3], and $\alpha \in (0,1)$ is the fractional derivative order.

3.1 Dimensional analysis of the model (6)

In Subsection 2.1 we saw that the Caputo derivative has dimension $T^{-\alpha}$. Furthermore, we saw that if f represents population (as is the case with functions S, I and R), then $cD_t^\alpha f(t)$ has dimension $\frac{P}{T^\alpha}$.

Knowing that each term given to the right side in the equations of the system (6) must have the same dimension as the term given to the left side (dimension $\frac{P}{T^\alpha}$), it is not right to include memory in the SIR model just by replacing the classical derivative with the fractional derivative, since the dimension of the terms on the right side does not correspond to that of the left side. For a correct dimensional analysis, one possibility is to raise the parameters β and γ to the value α. It is through this path that the formulation of the model (6) is obtained.

3.2 Greater efficiency of the fractional model and the memory effect from the order of the derivative

In [1, 5] the authors compare the classic version of the SIR model with the fractional version (with memory) to describe active cases of COVID-19 in some countries, using least squares data fit for both models. In this subsection, we recall the results obtained by them, to emphasize the advantage of using the Caputo derivative in epidemiological models, and we use these results as an example to identify the memory effect through the order α of the fractional derivative.

In [1, 5] it was shown that the fractional model is a good tool to study the phenomenon when there is a memory effect in the process. In Figures 3 and 4, we see the results obtained for the first "wave" of China and South Korea, where the actual data are represented by the blue dots, the solution of the classic model is given by the black continuous curve and the solution of the fractional model is given by the red dashed curve. Data were obtained from [10], since January 22, 2020 in the case of China and since February 15, 2020 in the case of South Korea.

These examples are enough to show that when there is a memory effect ($\alpha < 1$) the fractional model fits the actual data better than the classic model. Figure 3 (obtained with $\alpha = 0.7607$) shows this clearly and we can also prove it by calculating the mean squared error: in the classic case the error is approximately 0.1402 while in the fractional case it is approximately 0.0061.

We also chose the case of China, as we see in Figure 4, as it is a case where the fractional model did not fit data better than the classic one (but not worse either, since it generalizes the other), representing a scenario without effect of memory. The value $\alpha \sim 1$ in the case of China shows that the memory effect is almost nonexistent at this stage of the pandemic (first "wave"), that is, it shows that recent information was more significant than information from the past. This makes sense, since the isolation in China began at a time when nothing was known about the disease, as it is in this country where the first cases appeared. Thus, they had no information from the past, so memory could not yet be present.

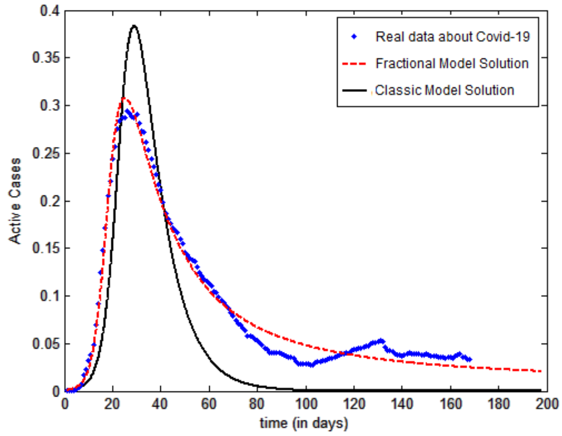

Figure 3: Solutions of the classic and fractional models for COVID-19 in South Korea. For the classic model, $\beta = 0.4047$ and $\gamma = 0.1071$. For the fractional model, $\beta = 0.5499$, $\gamma = 0.1055$ and $\alpha = 0.7607$.

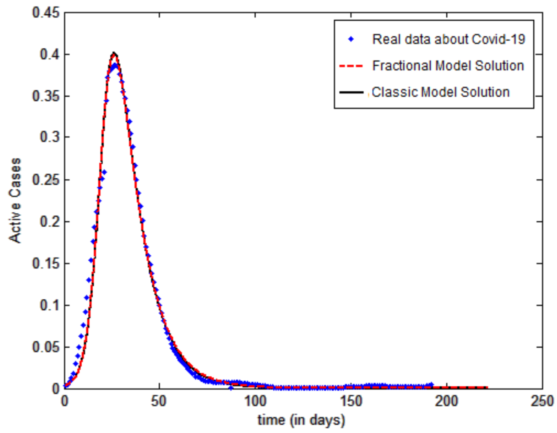

Figure 4: Solutions of the classic and fractional models for COVID-19 in China. For the classic model, $\beta = 0.3591$ and $\gamma = 0.0907$. For the fractional model, $\beta = 0.3613$, $\gamma = 0.0907$ and $\alpha = 0.9937$.

On the other hand, in South Korea there was already some information that could be used to take measures to contain the pandemic, based on what had already been experienced in other countries such as China. That is, the memory effect was already present.

Therefore, the smaller the value of α, the greater the memory effect, which illustrates the concept presented in Subsection 2.2. Moreover, as it is a scenario with memory ($\alpha < 1$), we see that the fractional model is more efficient than the classic model.

4 Final Considerations

Fractional calculus provides good tools for mathematical modeling because of its ability to include memory effects in dynamics. This fact is explicitly proven by Equation (2) for the case of Caputo fractional derivative, where we see that this operator is proportional to the weighted average of historical values of the classical derivative.

Furthermore, the Equation (2) allows us to conclude more interesting facts:

- Equation (4) (obtained through Equation (2)) shows the dimension of the Caputo derivative;
- Through the probability density function of the weighted average that describes the Caputo derivative, it is possible to observe which historical values of the classical derivative have greater weight to obtain the current instant, according to the value of $\alpha \in (0, 1)$;
- The Equation (2) shows that the Caputo derivative is not zero at the critical point and, even more, shows that when it is zero (in t_α) the critical point (in t^*) has already occurred;
- From the comments above, we note that the difference between the value of t^α and t^* increases as α decreases, and the value α may be related to the efficacy of the measures adopted. The lower the α is, the less effective these measures are, and the past occurrences interfere with the current dynamics of the disease for longer.

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