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Calculation of Green's Function for Poisson's Equation in Plane Polar Coordinates using Eigenfunction Expansion in the Angular Variable

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Abstract. A new calculation of Green's function for the problem with Poisson's equation in plane polar coordinates is presented. The method consists in calculating the solution of a problem that is simpler but that has the same Green's function – the problem that results from the homogenization of the boundary conditions – and then inferring Green's function by comparing this calculated solution with Green's formula for the solution. To describe the method, it is applied to the particular case of a disc sector under mixed Dirichlet-Neumann boundary conditions. The solution of the simplified problem is obtained as an eigenfunction expansion in the angular variable. Green's function arises from the calculations as an infinite series but is finally presented in closed form because it is possible to compute the sum of this series.

Keywords. Green's function, Poisson, plane polar coordinates, closed form, Dirichlet, Neumann.

1 Introduction

This work aims to describe a new method for calculating Green's function for a boundary value problem based on the Poisson's equation expressed in the plane polar coordinates and to present it in closed form. To explain the method, we consider the domain Ω of the problem to be the disc sector shown in Figure 1 as well as the boundary conditions to be those indicated there: Dirichlet's on the rectilinear boundary along the x-axis and on the circular boundary, and Neumann's on the other rectilinear boundary. This problem is formulated as follows:

$$\begin{cases} \nabla^2 u(r,\theta) = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = h(r,\theta) \text{ with } r \in (0,b) \text{ and } \theta \in (0,\gamma) \ , \\ u(r,0) = f_0(r) \text{ if } r \in [0,b) \ , \ \ \dfrac{\partial u}{\partial \theta}(r,\gamma) = g_\gamma(r) \text{ if } r \in (0,b) \ , \ \ u(b,\theta) = f_b(\theta) \text{ if } \theta \in [0,\gamma] \ . \end{cases} \tag{1}$$

where the functions f_0, f_b, g_γ, and h are continuous, and we want a continuous u in $\Omega \cup \partial\Omega$.

Section 2 describes the main steps of the method. Section 3 presents the application of the method to calculate Green's function for problem (1). Section 4 shows how to compute the sum of the infinite series in the Green's function expression calculated in Section 3 to finally present it in closed form. Section 5 contains a comparison of this closed-form Green's function with the one provided by the method of images for the particular case when $\gamma = \pi/2$. Section 6 ends the body of the paper with final comments.

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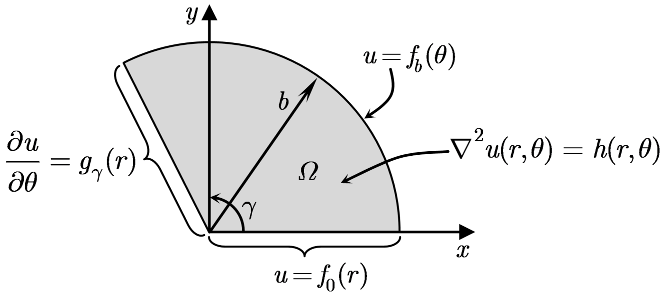

Figure 1: The problem solved in this work: that in (1).

2 Description of the Main Steps of the Method

The method developed in this work takes advantage of the fact that Green's function of problem (1) does not depend on the functions f_0, f_b, g_γ, and h [3, sec. 2.2.4]. Then, to calculate it, we consider the following simplified version of the problem, in which all boundary conditions are homogenized:

$$\begin{cases} \nabla^2 v(r,\theta) = \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = h(r,\theta) \text{ with } r \in (0,b)\ \theta \in (0,\gamma)\ , \\ v(r,0) = 0 \text{ if } r \in [0,b)\ ,\ \dfrac{\partial v}{\partial \theta}(r,\gamma) = 0 \text{ if } r \in (0,b)\ ,\ v(b,\theta) = 0 \text{ if } \theta \in [0,\gamma]\ . \end{cases} \tag{2}$$

a) The first step of the method is the calculation of the solution v of the above problem. In this work, we solve it by the method of eigenfunction expansion. We write

$$v(r,\theta) = \sum_{n=1,3,5\cdots} v_n(r) \sin \frac{n\pi\theta}{2\gamma}\ , \tag{3}$$

that is, we admit that the solution to problem (2) can be expanded into the eigenfunctions $\Theta_n(\theta) = \sin(n\pi\theta/2\gamma)$ $(n = 1, 3, 5 \cdots)$ [1, sec. 10.1, Prob. 19, p. 595 & 786] that arise when the separation of variables $v(r,\theta) = R(r)\Theta(\theta)$ is used to solve problem (2) in the particular case where $h(r,\theta) \equiv 0$ (Laplace's equation) and the homogeneous condition on the circular boundary (at $r = b$) is replaced by a *nonhomogeneous* one. Notice that, as a consequence of the chosen eigenfunctions, (3) automatically satisfies the conditions of problem (2) on the boundaries at $\theta = 0$ and $\theta = \gamma$.

With the substitution of (3) into the partial differential equation (PDE) and into the boundary condition in (2) that remains to be satisfied (that at $r = b$) one obtains a one-dimensional boundary value problem for $v_n(r)$: an nonhomogeneous ordinary differential equation (ODE) under a homogeneous Dirichlet condition. After solving this one-dimensional problem for $v_n(r)$, we substitute its solution into (3) to complete the calculation of the solution $v(r,\theta)$ to problem (2).

b) The second step is the determination of Green's function $G(\vec{r}\,|\vec{r}\,') = G(r,\theta\,|\,r',\theta')$ for problem (1) from the solution $v(\vec{r}) = v(r,\theta)$ of problem (2) calculated in the previous step. This is done as follows: Since Green's formula for the solution $v(r,\theta)$ to problem (2) is simply given by {cf. Ref. [5], eq. (1.42), which is here adapted to two dimensions}

$$v(r,\theta) = -\frac{1}{2\pi}\iint_\Omega G(\vec{r}\,|\vec{r}\,')\,h(\vec{r}\,')\,dA' = -\frac{1}{2\pi}\int_0^\gamma d\theta' \int_0^b dr'\, r'\, h(r',\theta')\, G(r,\theta\,|\,r',\theta')\ , \tag{4}$$

it is possible to infer an expression for $G(r,\theta\,|\,r',\theta')$ by writing the already calculated solution $v(r,\theta)$ in the form of the double integral on the right side of this equation. We will see that this writing is not an automatic task, requiring some artifices in the first step.

c) A third step is still necessary, because the Green's function expression obtained in the second step still involves an infinite series. We need, therefore, to evaluate the sum of this series to obtain Green's function in closed form.

3 Application of the Method

Now we apply the method to calculate Green's function of problem (1). The first step is to solve problem (2) by substituting (3) into the PDE of this problem; we get

$$\sum_{n=1,3,5\cdots}\left[v_n'' + \frac{1}{r}v_n' - \frac{(n\pi/2\gamma)^2}{r^2}v_n\right]\sin\frac{n\pi\theta}{2\gamma} = h \ .$$

This result shows that the terms in brackets for $n = 1, 3, 5\cdots$ are the coefficients of a generalized Fourier sine series of the function h; therefore,

$$v_n'' + \frac{1}{r}v_n' - \frac{(n\pi/2\gamma)^2}{r^2}v_n(r) \ = \ \frac{2}{\gamma}\int_0^\gamma h(r,\theta)\sin\frac{n\pi\theta}{2\gamma}\,d\theta \ \equiv \ h_n(r) \ . \tag{5}$$

We thus see that $v_n(r)$ is the solution of a nonhomogeneous Euler-Cauchy ODE [4, sec. 1.6].

Since the general solution of the associated homogeneous equation is

$$v_{Hn}(r) = c_{1n}r^{n\pi/2\gamma} + c_{2n}/r^{n\pi/2\gamma} \ ,$$

a particular solution by the method of variation of parameters [4, sec. 1.9] has the form

$$v_{Pn}(r) = A_n(r)r^{n\pi/2\gamma} + B_n(r)/r^{n\pi/2\gamma} \ , \tag{6}$$

where the functions $A_n(r)$ and $B_n(r)$ are solutions of the system of equations

$$\begin{cases} A_n' r^{n\pi/2\gamma} + B_n'/r^{n\pi/2\gamma} = 0 \\ (n\pi/2\gamma)A_n' r^{(n\pi/2\gamma)-1} - (n\pi/2\gamma)B_n'/r^{(n\pi/2\gamma)+1} = h_n \ . \end{cases}$$

Solving it, we get

$$A_n'(r) = \frac{\gamma h_n(r)}{n\pi r^{(n\pi/2\gamma)-1}} \quad \Rightarrow \quad A_n(r) = \frac{\gamma}{n\pi}\int_0^r \frac{h_n(r')}{r'^{(n\pi/2\gamma)-1}}dr' \ ,$$

$$B_n'(r) = -\frac{\gamma h_n(r)\, r^{(n\pi/2\gamma)+1}}{n\pi} \quad \Rightarrow \quad B_n(r) = -\frac{\gamma}{n\pi}\int_0^r h_n(r')\, r'^{(n\pi/2\gamma)+1}dr' \ .$$

Using these results in (6), we can write the general solution $v_{Hn}(r) + v_{Pn}(r)$ of (5) as

$$v_n(r) = \left[c_{1n} + \frac{\gamma}{n\pi}\int_0^r \frac{h_n(r')}{r'^{(n\pi/2\gamma)-1}}dr'\right]r^{n\pi/2\gamma} + \left[c_{2n} - \frac{\gamma}{n\pi}\int_0^r h_n(r')\, r'^{(n\pi/2\gamma)+1}dr'\right]\frac{1}{r^{n\pi/2\gamma}} \ . \tag{7}$$

To determine c_{1n} and c_{2n}, we impose the conditions of the problem, first the one related to continuity. To prevent (7) from becoming infinite when $r \to 0$, it is necessary that

$$\lim_{r\to 0}\left[c_{2n} - \frac{\gamma}{n\pi}\int_0^r h_n(r')\, r'^{(n\pi/2\gamma)+1}dr'\right] = 0 \quad \Rightarrow \quad c_{2n} = 0 \ .$$

With this result, (7) becomes

$$v_n(r) = \left[c_{1n} + \frac{\gamma}{n\pi}\int_0^r \frac{h_n(r')}{r'^{(n\pi/2\gamma)-1}}dr'\right]r^{n\pi/2\gamma} + \left[-\frac{\gamma}{n\pi}\int_0^r h_n(r')\, r'^{(n\pi/2\gamma)+1}dr'\right]\frac{1}{r^{n\pi/2\gamma}} \ . \tag{8}$$

Now we require that (8) satisfies the condition

$$v_n(b) = 0 \ , \tag{9}$$

which results from the substitution of (3) into the condition $v(b,\theta) = 0$ given in (2). We obtain

$$c_{1n} = \frac{\gamma}{n\pi}\int_0^b dr' r' h_n(r')\left[\left(\frac{r'}{b^2}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{1}{r'}\right)^{\frac{n\pi}{2\gamma}}\right] \ .$$

Using this expression for c_{1n} in (8), we can write $v_n(r)$ as follows:

$$v_n(r) = \frac{\gamma}{n\pi}\int_0^b dr' r' h_n(r')\left[\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r}{r'}\right)^{\frac{n\pi}{2\gamma}}\right] + \frac{\gamma}{n\pi}\int_0^r dr' r' h_n(r')\left[\left(\frac{r}{r'}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r'}{r}\right)^{\frac{n\pi}{2\gamma}}\right] \ . \tag{10}$$

This expression of $v_n(r)$ is not suitable to express $v(r,\theta)$ in the form of the double integral in (4), because, in that double integral, the interval of integration with respect to r' is $[0,b]$, whereas, in the second integral above, it is $[0,r]$. To overcome this difficulty we derive another expression of $v_n(r)$ with a slightly different form as follows: Since the lower limit of integration of the indefinite integrals in (7) is an arbitrary point of $[0,b]$, we choose it now to be b (instead of 0) to obtain the following equivalent expression for the general solution of (5):

$$v_n(r) = \left[d_{1n} + \frac{\gamma}{n\pi}\int_b^r \frac{h_n(r')}{r'^{(n\pi/2\gamma)-1}}dr'\right] r^{n\pi/2\gamma} + \left[d_{2n} - \frac{\gamma}{n\pi}\int_b^r h_n(r') r'^{(n\pi/2\gamma)+1} dr'\right]\frac{1}{r^{n\pi/2\gamma}} \ . \tag{11}$$

As before, to prevent this expression from becoming infinite as $r \to 0$, it is necessary that

$$\lim_{r\to 0}\left[d_{2n} - \frac{\gamma}{n\pi}\int_b^r h_n(r') r'^{(n\pi/2\gamma)+1} dr'\right] = 0 \quad \Rightarrow \quad d_{2n} = \frac{\gamma}{n\pi}\int_b^0 h_n(r') r'^{(n\pi/2\gamma)+1} dr' \ ,$$

and, by imposing condition (9) on (11) and then substituting the above expression for d_{2n}, we get

$$v_n(b) = d_{1n} b^{n\pi/2\gamma} + \frac{d_{2n}}{b^{n\pi/2\gamma}} = 0 \quad \Rightarrow \quad d_{1n} = -\frac{\gamma}{n\pi (b^2)^{n\pi/2\gamma}}\int_b^0 h_n(r') r'^{(n\pi/\gamma)+1} dr' \ .$$

Using these results for d_{1n} and d_{2n} in (11), we can write the following expression for $v_n(r)$:

$$v_n(r) = \frac{\gamma}{n\pi}\int_0^b dr' r' h_n(r')\left[\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r'}{r}\right)^{\frac{n\pi}{2\gamma}}\right] + \frac{\gamma}{n\pi}\int_r^b dr' r' h_n(r')\left[\left(\frac{r'}{r}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r}{r'}\right)^{\frac{n\pi}{2\gamma}}\right] \ . \tag{12}$$

Now we have (10) and (12) expressing $v_n(r)$. The idea is to add these two equations and then replace the sum of the integrals $\int_0^r dr'$ and $\int_r^b dr'$ with the single integral $\int_0^b dr'$, whose interval of integration is the one in (4). Note, however, that the integrands of these two integrals are not exactly the same; but since one becomes the other by replacing r/r' with r'/r, one way to make these two integrals display the same integrand is to define

$$r_< \ (r_>) \equiv \text{the smaller (larger) of } r \text{ and } r' \ . \tag{13}$$

In fact, with this notation, because $r = r_>$ and $r' = r_<$ in the integral $\int_0^r dr'$, and $r = r_<$ and $r' = r_>$ in $\int_r^b dr'$, we have that (10) and (12) are respectively given by

$$v_n(r) = \frac{\gamma}{n\pi}\int_0^b dr' r' h_n(r')\left[\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r}{r'}\right)^{\frac{n\pi}{2\gamma}}\right] + \frac{\gamma}{n\pi}\int_0^r dr' r' h_n(r')\left[\left(\frac{r_>}{r_<}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}}\right]$$

and

$$v_n(r) = \frac{\gamma}{n\pi}\int_0^b dr' r' h_n(r')\left[\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r'}{r}\right)^{\frac{n\pi}{2\gamma}}\right] + \frac{\gamma}{n\pi}\int_r^b dr' r' h_n(r')\left[\left(\frac{r_>}{r_<}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}}\right] .$$

Therefore, adding these two equations, and again using (13), we get the proper form of the expression of $v_n(r)$ to be used in (3):

$$\begin{aligned}
2\,v_n(r) &= \frac{\gamma}{n\pi}\int_0^b dr'r'h_n(r')\left[2\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}} - \cancel{\left(\frac{r_>}{r_<}\right)^{\frac{n\pi}{2\gamma}}}\right] \\
&\quad + \frac{\gamma}{n\pi}\int_0^b dr'r'h_n(r')\left[\cancel{\left(\frac{r_>}{r_<}\right)^{\frac{n\pi}{2\gamma}}} - \left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}}\right] , \\
\Rightarrow \quad v_n(r) &= \frac{\gamma}{n\pi}\int_0^b dr'r'h_n(r')\left[\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}}\right] .
\end{aligned}$$

Taking this result into (3) and using the defining expression for h_n given in (5), we get

$$\begin{aligned}
v(r,\theta) &= \sum_{n\,=\,1,3,5\cdots}\left\{\frac{\gamma}{n\pi}\int_0^b dr'r'\left(\frac{2}{\gamma}\int_0^\gamma d\theta' h(r',\theta')\sin\frac{n\pi\theta'}{2\gamma}\right)\left[\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}}\right]\right\}\sin\frac{n\pi\theta}{2\gamma} \\
&= -\frac{1}{2\pi}\int_0^\gamma d\theta'\int_0^b dr'r'h(r',\theta')\underbrace{\sum_{n\,=\,1,3,5\cdots}\frac{4}{n}\left[\left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}}\right]\sin\frac{n\pi\theta'}{2\gamma}\sin\frac{n\pi\theta}{2\gamma}}_{G(r,\theta\,|\,r',\theta')} ,
\end{aligned}$$

from which, by comparison with (4), we infer the expression for $G(r,\theta\,|\,r',\theta')$ indicated above:

$$G(r,\theta\,|\,r',\theta') = \sum_{n\,=\,1,3,5\cdots}\frac{4}{n}\left[\left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}}\right]\sin\frac{n\pi\theta'}{2\gamma}\sin\frac{n\pi\theta}{2\gamma} \ . \tag{14}$$

4 Green's Function in Closed Form

To derive Green's function in closed form we need to calculate the sum of the infinite series in (14). To this end, using the definition $z \equiv p\,e^{\mathrm{i}\varphi}$, from which $z^n = p^n e^{\mathrm{i}n\varphi} = p^n\cos n\varphi + \mathrm{i}p^n\sin n\varphi$, we first evaluate the sum of the following infinite series:

$$\begin{aligned}
&\sum_{n\,=\,1,3,5\cdots}\frac{1}{n}\underbrace{p^n\cos n\varphi}_{\mathrm{Re}\,z^n} = \mathrm{Re}\sum_{k\,=\,0}^{\infty}\frac{z^{2k+1}}{2k+1} = \mathrm{Re}\int_0^z\left[\sum_{k\,=\,0}^{\infty}(\zeta^2)^k\right]d\zeta = \mathrm{Re}\int_0^z\frac{1}{1-\zeta^2}\,d\zeta = \\
&\mathrm{Re}\left[\frac{1}{2}\ln(z+1) - \frac{1}{2}\ln(z-1)\right] = \frac{1}{2}\ln|z+1| - \frac{1}{2}\ln|z-1| = \frac{1}{4}\ln\frac{1+p^2+2p\cos\varphi}{1+p^2-2p\cos\varphi} \quad (0 \le p < 1) ,
\end{aligned}$$

where, to find the sum of the infinite series between square brackets, we used the formula for the sum of a geometric series (noticing that $|\zeta^2| \le |z|^2 = p^2 < 1$ along the straight path of integration from $\zeta = 0$ to $\zeta = z$), and we also considered the definition of the complex logarithm.

Now, using the result above, we deduce the sum of this other infinite series:

$$\begin{aligned}
S &\equiv \sum_{n\,=\,1,3,5\cdots}\frac{4}{n}\left(\frac{A}{B}\right)^{\frac{n\pi}{2\gamma}}\sin\frac{n\pi\theta'}{2\gamma}\sin\frac{n\pi\theta}{2\gamma} = \sum_{n\,=\,1,3,5\cdots}\frac{2}{n}p^n\left[\cos\tfrac{n\pi(\theta'-\theta)}{2\gamma} - \cos\tfrac{n\pi(\theta'+\theta)}{2\gamma}\right]\Bigg|_{p\,=\,(\frac{A}{B})^{\pi/2\gamma}} \\
&= \left[\frac{1}{2}\ln\frac{1+p^2+2p\cos\frac{\pi(\theta'-\theta)}{2\gamma}}{1+p^2-2p\cos\frac{\pi(\theta'-\theta)}{2\gamma}} - \frac{1}{2}\ln\frac{1+p^2+2p\cos\frac{\pi(\theta'+\theta)}{2\gamma}}{1+p^2-2p\cos\frac{\pi(\theta'+\theta)}{2\gamma}}\right]_{p\,=\,(\frac{A}{B})^{\pi/2\gamma}} \\
&= \frac{1}{2}\ln\frac{A^{\frac{\pi}{\gamma}} + B^{\frac{\pi}{\gamma}} + 2(AB)^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}}{A^{\frac{\pi}{\gamma}} + B^{\frac{\pi}{\gamma}} - 2(AB)^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}} - \frac{1}{2}\ln\frac{A^{\frac{\pi}{\gamma}} + B^{\frac{\pi}{\gamma}} + 2(AB)^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}}{A^{\frac{\pi}{\gamma}} + B^{\frac{\pi}{\gamma}} - 2(AB)^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}} .
\end{aligned}$$

Finally, since $G(r,\theta\,|\,r',\theta') = S\Big|_{\substack{A=r_<\\B=r_>}} - S\Big|_{\substack{A=rr'/b\\B=b}}$, as wee see from (14), we have that

$$G(r,\theta\,|\,r',\theta') \;=\; \frac{1}{2}\,\ln\frac{r^{\frac{\pi}{\gamma}}+r'^{\frac{\pi}{\gamma}}+2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}}{r^{\frac{\pi}{\gamma}}+r'^{\frac{\pi}{\gamma}}-2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}} - \frac{1}{2}\,\ln\frac{r^{\frac{\pi}{\gamma}}+r'^{\frac{\pi}{\gamma}}+2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}}{r^{\frac{\pi}{\gamma}}+r'^{\frac{\pi}{\gamma}}-2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}}$$
$$-\frac{1}{2}\,\ln\frac{(rr'/b)^{\frac{\pi}{\gamma}}+b^{\frac{\pi}{\gamma}}+2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}}{(rr'/b)^{\frac{\pi}{\gamma}}+b^{\frac{\pi}{\gamma}}-2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}} + \frac{1}{2}\,\ln\frac{(rr'/b)^{\frac{\pi}{\gamma}}+b^{\frac{\pi}{\gamma}}+2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}}{(rr'/b)^{\frac{\pi}{\gamma}}+b^{\frac{\pi}{\gamma}}-2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}} \;. \tag{15}$$

5 Comparison with the Solution Given by the Method of Images when the Domain is the First Quadrant of the Disc

The method of images {cf. Refs. [7, sec. VII.13] and [2, sec. 3]} allows to obtain the solution faster when the problem presents a symmetry that allows to quickly infer the configuration of images to use. Let us then apply this method for the particular case in which $\gamma=\pi/2$ to check (15). In this case, we need the seven images P_1^-, P_2^+, P_3^-, P_4^-, P_5^+, P_6^-, and P_7^+ shown in Figure 2, in which the superscript $+$ or $-$ indicates that the corresponding harmonic term has a $+1$ or -1 multiplying it. These terms are of the form [7, sec. VII.13, last paragraph] $\pm\ln(1/|\vec r\,'-\vec r_n|)$, if the corresponding image is generated by reflection with respect to the x or y-axis, or $\pm\ln\big[(b/r)/|\vec r\,'-\vec r_n|\big]$, if by inversion with respect to the circle of radius b centered at the origin, where $\vec r_n$ denotes the position vetor of the n-th image. Green's function is therefore given by

$$G(\vec r\,|\,\vec r\,') = \ln\frac{1}{|\vec r\,'-\vec r|} - \ln\frac{1}{|\vec r\,'-\vec r_1|} + \ln\frac{1}{|\vec r\,'-\vec r_2|} - \ln\frac{b/r}{|\vec r\,'-\vec r_3|}$$
$$-\ln\frac{1}{|\vec r\,'-\vec r_4|} + \ln\frac{b/r}{|\vec r\,'-\vec r_5|} - \ln\frac{b/r}{|\vec r\,'-\vec r_6|} + \ln\frac{b/r}{|\vec r\,'-\vec r_7|} \;. \tag{16}$$

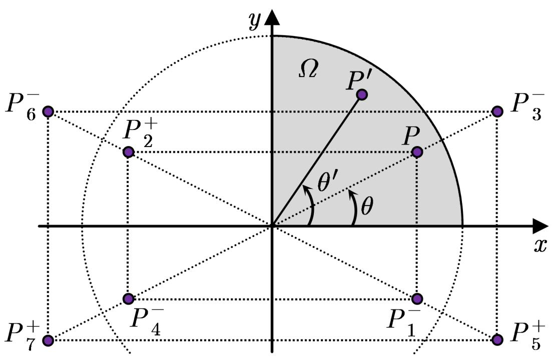

Figure 2: The configuration of images used to get, by the method of images, Green's function for problem (1) when $\gamma=\pi/2$ (the case in which the domain Ω is the first quadrant of the disc).

The plane polar coordinates of the position vectors above are as follows:

$$\vec r\,(r\,,\,\theta)\,,\quad \vec r\,'(r'\,,\,\theta')\,,\quad \vec r_1(r\,,\,2\pi-\theta)\,,\quad \vec r_2(r\,,\,\pi-\theta)\,,\quad \vec r_3(b^2/r\,,\,\theta)\,,$$
$$\vec r_4(r\,,\,\pi+\theta)\,,\quad \vec r_5(b^2/r\,,\,2\pi-\theta)\,,\quad \vec r_6(b^2/r\,,\,\pi-\theta)\,,\quad \vec r_7(b^2/r\,,\,\pi+\theta)\;.$$

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Therefore, by using the definition of magnitude of a vector, or, geometrically, the law of cosines, we can calculate all the distances $|\vec{r}\,' - \vec{r}\,|$, $|\vec{r}\,' - \vec{r}_1|$, $|\vec{r}\,' - \vec{r}_2| \cdots$ in (16), obtaining

$$\begin{aligned} G(r,\theta|r',\theta') = &-(1/2)\ln\left[r^2 + {r'}^2 - 2rr'\cos(\theta'-\theta)\right] + (1/2)\ln\left[r^2 + {r'}^2 - 2rr'\cos(\theta'+\theta)\right] \\ &- (1/2)\ln\left[r^2 + {r'}^2 + 2rr'\cos(\theta'+\theta)\right] + (1/2)\ln\left[(rr'/b)^2 + b^2 - 2rr'\cos(\theta'-\theta)\right] \\ &+ (1/2)\ln\left[r^2 + {r'}^2 + 2rr'\cos(\theta'-\theta)\right] - (1/2)\ln\left[(rr'/b)^2 + b^2 - 2rr'\cos(\theta'+\theta)\right] \\ &+ (1/2)\ln\left[(rr'/b)^2 + b^2 + 2rr'\cos(\theta'+\theta)\right] - (1/2)\ln\left[(rr'/b)^2 + b^2 + 2rr'\cos(\theta'-\theta)\right] , \end{aligned}$$

which is exactly the same Green's function given by (15) with $\gamma = \pi/2$.

6 Final Comments

To get (4) from the cited eq. (1.42) in Ref. [5], we need to multiply this equation by $4\pi/2\pi$, ignore the integral over the boundary, and replace $-\rho/\epsilon_0$ with h, because our problem (2) is two-dimensional (and not tridimensional), has homogeneous boundary conditions, and exhibits simply h (instead of $-\rho/\epsilon_0$) in the right-hand side of Poisson's equation.

The method can be applied in any domain Ω where $r \in (a,b)$ and $\theta \in (0,\gamma)$, with any a and b such that $b > a \geq 0$ and any $\gamma \in (0, 2\pi]$. Furthermore, many experimental calculations performed privately indicate that, whenever the boundary conditions are Dirichlet's or Neumann's, it will be possible to determine Green's function in *closed form*, which is a nice feature of the method.

When there are enough symmetries, the method of images also gives results in closed form and, moreover, more quickly, thus becoming the best method. But when this is not the case (for $\gamma \neq \pi/2$, π, $3\pi/2$, 2π), this method becomes considerably involved, and the method presented here is preferable.

The problem considered in this work finds application in fields such as Newtonian gravity, electrostatics as well as fluid dynamics (cf. the pressure Poisson equation in Ref. [6], sec. 2).

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