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The Development of the Multiscale Hybrid-Hybrid-Mixed method

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Abstract. This work develops the Multiscale Hybrid-Hybrid Mixed method - MH^{2}M. This is a finite element method that efficiently solves elliptic partial differential equations with multiscale heterogeneous coefficients. The starting point is the Three-field domain decomposition formulation, which searches a function, defined within each subdomain, and two Lagrange multipliers: the flow and trace of the function posed on interfaces. This setting allows different discretizations in each subdomain, as well as the use of different numerical methods to solve local problems. After the decomposition of functional spaces and two static condensations, the MH^{2}M method arises by solving independent local Neumann problems in parallel. It results that the method solves an elliptic global problem posed at interfaces instead of the more complicated three-field formulation. In addition to the lower computational cost, the use of iterative methods such as the conjugate gradient is possible. A proper compatibility condition enables a discretization using non-matching grids, preserving stability. Finally, we establish error estimates for a pair of compatible finite element spaces.

Key words. Numerical methods, Finite element methods, Multiscale Hybrid-Hybrid-Mixed method, Multiscale Hybrid-Mixed method.

1 Setting and preliminaries results

1.1 The model problem

Let Ω be an open bounded subset of $\mathbb{R}^d$, $d = 2, 3$, with a polygonal boundary $\partial\Omega$. The model problem consists of finding $u : \Omega \to \mathbb{R}$, weak solution of

$$\begin{aligned} -\nabla \cdot (\mathcal{A}\nabla u) &= f, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where $f \in L^2(\Omega)$. The coefficient matrix $\mathcal{A} \in [L^\infty(\Omega)]^{d\times d}$ is symmetric and there are constants $0 < a_{\min} < a_{\max}$ such that the following ellipticity condition holds

$$a_{\min}|\mathbf{v}|^2 \le \mathcal{A}(\mathbf{x})\mathbf{v}\cdot\mathbf{v} \le a_{\max}|\mathbf{v}|^2, \tag{2}$$

for all $\mathbf{v} \in \mathbb{R}^d$ and almost everywhere $\mathbf{x} \in \Omega$. The corresponding variational formulation of (1) gives that $u \in \mathrm{H}^1_0(\Omega)$ satisfy

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla v \, d\mathbf{x} = \int_\Omega f v \, d\mathbf{x}, \quad \forall\, v \in \mathrm{H}^1_0(\Omega). \tag{3}$$

The Poincaré inequality and the Lax-Milgram Lemma [4] ensure that (3) is well-posed.

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1.2 Definitions

We next consider a partition of the domain Ω and define some suitable functional spaces to propose a different formulation for (3). Let $\mathcal{T}_{\mathcal{H}} := \{K\}$ be a regular and conforming triangulation of Ω, with characteristic length $\mathcal{H} \in]0,1[$. The intersection of two different elements K_i, $K_j \in \mathcal{T}_{\mathcal{H}}$ is either empty, a node or a face. In particular each face F is such that $F \subset \partial K_i \cap \partial K_j$ for some two elements, or $F \in \partial\Omega$. We denote the skeleton of the mesh $\mathcal{T}_{\mathcal{H}}$ by $\mathcal{E}_{\mathcal{H}}$, given by the set of all faces F. Let $\mathbf{n}$ the exterior normal vector on $\partial\Omega$ and, for each $K \in \mathcal{T}_{\mathcal{H}}$, let $\mathbf{n}^K$ be the unitary outward normal vector on ∂K.

Next, consider the following broken functional spaces:

$$\mathrm{H}^1(\mathcal{T}_{\mathcal{H}}) := \left\{v \in L^2(\Omega);\ v|_K \in \mathrm{H}^1(K),\ K \in \mathcal{T}_{\mathcal{H}}\right\}, \quad \Lambda := \prod_{K\in\mathcal{T}_{\mathcal{H}}} \mathrm{H}^{-1/2}(\partial K), \tag{4}$$

$$\mathrm{H}_0^{1/2}(\mathcal{E}_{\mathcal{H}}) := \left\{v|_{\mathcal{E}_{\mathcal{H}}};\ v \in \mathrm{H}_0^1(\Omega)\right\}. \tag{5}$$

We denote by $\mathrm{H}^{1/2}(\partial K)$ the space of traces on ∂K of functions in $H^1(K)$, and by $\mathrm{H}^{-1/2}(\partial K)$ its dual space. For $w, v \in \mathrm{H}^1(\mathcal{T}_{\mathcal{H}})$, $\rho \in \mathrm{H}_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ and $\mu \in \Lambda$, let:

$$(v,w)_{\mathcal{T}_{\mathcal{H}}} := \sum_{K\in\mathcal{T}_{\mathcal{H}}} \int_K wv\,dx, \quad \langle\mu,\rho\rangle_{\mathcal{E}_{\mathcal{H}}} = \sum_{K\in\mathcal{T}_{\mathcal{H}}} \langle\mu,\rho\rangle_{\partial K}, \tag{6}$$

where $\langle\cdot,\cdot\rangle_{\partial K}$ is the dual product involving $\mathrm{H}^{-1/2}(\partial K)$ and $\mathrm{H}^{1/2}(\partial K)$.

1.3 The Three-field domain decomposition method

At this point, we proceed to develop the three-field variational formulation of (3). It follows from (1) that $\lambda := \mathcal{A}\nabla u \cdot \mathbf{n} \in \mathrm{H}^{-1/2}(\mathcal{E}_{\mathcal{H}})$. Thus, λ is the trace of a function in $\mathrm{H}(\mathrm{div};\Omega)$. From [4, Lemma 3.4], that is equivalent to $\lambda \in \Lambda$ and

$$\langle\lambda,\xi\rangle_{\mathcal{E}_{\mathcal{H}}} = 0 \tag{7}$$

for all $\xi \in \mathrm{H}_0^{1/2}(\mathcal{E}_{\mathcal{H}})$. On the other hand, consider $\rho \in \mathrm{H}_0^{1/2}(\mathcal{E}_{\mathcal{H}})$. The Hahn-Banach Theorem [2, Theorem $5.9-3$] allows us to impose ρ as the trace of u over $\mathcal{E}_{\mathcal{H}}$, that is,

$$\langle\mu, u-\rho\rangle_{\mathcal{E}_{\mathcal{H}}} = 0, \tag{8}$$

for all $\mu \in \Lambda$. Therefore, assuming that (3) holds, we gather (3), (7) and (8) and obtain the Three-field formulation: find $u \in \mathrm{H}^1(\mathcal{T}_{\mathcal{H}})$, $\rho \in \mathrm{H}_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ and $\lambda \in \Lambda$ such that

$$\begin{array}{llll} (\mathcal{A}\nabla u,\nabla v)_{\mathcal{T}_{\mathcal{H}}} & -\langle\lambda,v\rangle_{\mathcal{E}_{\mathcal{H}}} & & = (f,v)_{\mathcal{T}_{\mathcal{H}}}, \quad \forall\, v \in \mathrm{H}^1(\mathcal{T}_{\mathcal{H}}); \\ -\langle\mu,u\rangle_{\mathcal{E}_{\mathcal{H}}} & & +\langle\mu,\rho\rangle_{\mathcal{E}_{\mathcal{H}}} & = 0, \quad \forall\,\mu \in \Lambda; \\ & \langle\lambda,\xi\rangle_{\mathcal{E}_{\mathcal{H}}} & & = 0, \quad \forall\,\xi \in \mathrm{H}_0^{1/2}(\mathcal{E}_{\mathcal{H}}). \end{array} \tag{9}$$

Remark 1.1. *Note that it is possible to recover the infinite dimensional formulation of the Multiscale Hybrid-Mixed method (MHM) [5]. Let the jump operator $[\![\cdot]\!] : \Lambda \to L^2(\mathcal{E}_{\mathcal{H}})$ such that, if $\tau \in \Lambda$, then $[\![\tau]\!]|_F := \tau|_F \cdot \boldsymbol{n}^{K_i} + \tau|F \cdot \boldsymbol{n}^{K_j}$ on a shared face $F \in K_i \cap K_j$. Consider the subspace*

$$\Lambda^* := \{\mu \in \Lambda;\ [\![\mu]\!]|_F = 0,\ \forall\, F \in \mathcal{E}_{\mathcal{H}} \setminus \partial\Omega\} \subset \Lambda.$$

By replacing Λ with Λ^* in (9), the terms $\langle\lambda,\xi\rangle_{\mathcal{E}_{\mathcal{H}}}$ and $\langle\mu,\rho\rangle_{\mathcal{E}_{\mathcal{H}}}$ vanish, for all $\mu \in \Lambda^*$ and all $\xi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$. Thus, the problem becomes: find a pair $(u,\lambda) \in H^1(\mathcal{T}_{\mathcal{H}}) \times \Lambda^*$ such that

$$\begin{array}{lll} (\mathcal{A}\nabla u,\nabla v)_{\mathcal{T}_{\mathcal{H}}} & -\langle\lambda,v\rangle_{\mathcal{E}_{\mathcal{H}}} & = (f,v)_{\mathcal{T}_{\mathcal{H}}}, \quad \forall\, v \in H^1(\mathcal{T}_{\mathcal{H}}); \\ -\langle\mu,u\rangle_{\mathcal{E}_{\mathcal{H}}} & & = 0, \quad \forall\,\mu \in \Lambda^*. \end{array}$$

1.4 Infinite dimensional global problem

Now, we perform space decomposition and static condensations to make problem (9) simpler to solve. Let us decompose both spaces $\mathrm{H}^1(\mathcal{T}_\mathcal{H})$ and Λ in the form "constant" plus "zero average", i.e.

$$\mathrm{H}^1(\mathcal{T}_\mathcal{H}) = \mathbb{P}_0(\mathcal{T}_\mathcal{H}) \oplus \tilde{\mathrm{H}}^1(\mathcal{T}_\mathcal{H}), \quad \text{and} \quad \Lambda := \Lambda^0 \oplus \tilde{\Lambda}, \tag{10}$$

where $\mathbb{P}_0(\mathcal{T}_\mathcal{H})$ is the space of piecewise constants in each element and

$$\tilde{\mathrm{H}}^1(\mathcal{T}_\mathcal{H}) := \left\{ v \in \mathrm{H}^1(\mathcal{T}_\mathcal{H});\ \int_{\partial K} v\, ds = 0,\ \forall\, K \in \mathcal{T}_\mathcal{H} \right\}; \tag{11}$$

$$\Lambda^0 := \operatorname{span} \left\{ \mu^0 \in \Lambda; \langle \mu^0, v\rangle_{\partial K} := \int_{\partial K} v\, ds,\ v \in \mathrm{H}^1(\mathcal{T}_\mathcal{H}),\ K \in \mathcal{T}_\mathcal{H} \right\}; \tag{12}$$

$$\tilde{\Lambda} := \Lambda \cap \mathbb{P}_0^\perp(\mathcal{T}_\mathcal{H}). \tag{13}$$

Then we can write $u = u^0 + \tilde{u}$, where $u^0 \in \mathbb{P}_0(\mathcal{T}_\mathcal{H})$ and $\tilde{u} \in \tilde{\mathrm{H}}^1(\mathcal{T}_\mathcal{H})$, and also $\lambda = \lambda^0 + \tilde{\lambda}$, for $\lambda^0 \in \Lambda^0$ and $\tilde{\lambda} \in \tilde{\Lambda}$. Decomposition (10) implies in a pre-processing stage to find $\lambda^0 \in \Lambda^0$ for which

$$\langle \lambda^0, v^0\rangle_{\mathcal{E}_\mathcal{H}} = -(f, v^0)_{\mathcal{T}_\mathcal{H}}, \quad \forall\, v^0 \in \mathbb{P}_0(\mathcal{T}_\mathcal{H}); \tag{14}$$

Also, after computing $\rho \in \mathrm{H}_0^{1/2}(\mathcal{E}_\mathcal{H})$, there is a post-processing stage to get $u^0 \in \mathbb{P}_0(\mathcal{T}_\mathcal{H})$ such that

$$\langle \mu^0, u^0\rangle_{\mathcal{E}_\mathcal{H}} = \langle \mu^0, \rho\rangle_{\mathcal{E}_\mathcal{H}}, \quad \forall\, \mu^0 \in \Lambda^0. \tag{15}$$

It remains to solve the system with zero-mean functions: find $(\tilde{u}, \tilde{\lambda}, \rho) \in \tilde{\mathrm{H}}^1(\mathcal{T}_\mathcal{H}) \times \tilde{\Lambda} \times \mathrm{H}_0^{1/2}(\mathcal{E}_\mathcal{H})$ solution of

$$\begin{array}{llll} (\mathcal{A}\nabla\tilde{u}, \nabla\tilde{v})_{\mathcal{T}_\mathcal{H}} & -\langle\tilde{\lambda}, \tilde{v}\rangle_{\mathcal{E}_\mathcal{H}} & & = (f, \tilde{v})_{\mathcal{T}_\mathcal{H}}, & \forall\, \tilde{v} \in \tilde{\mathrm{H}}^1(\mathcal{T}_\mathcal{H}), \\ -\langle\tilde{\mu}, \tilde{u}\rangle_{\mathcal{E}_\mathcal{H}} & & +\langle\tilde{\mu}, \rho\rangle_{\mathcal{E}_\mathcal{H}} & = 0 & \forall\, \tilde{\mu} \in \tilde{\Lambda}, \\ & \langle\tilde{\lambda}, \xi\rangle_{\mathcal{E}_\mathcal{H}} & & = -\langle\lambda^0, \xi\rangle_{\mathcal{E}_\mathcal{H}}, & \forall\, \xi \in \mathrm{H}_0^{1/2}(\mathcal{E}_\mathcal{H}). \end{array} \tag{16}$$

The first equation consists in finding weak solutions of two local Neumann problems:

1. find $\mathrm{T}\tilde{\mu} \in \tilde{\mathrm{H}}^1(\mathcal{T}_\mathcal{H})$ such that

$$\begin{aligned} \nabla \cdot (\mathcal{A}\nabla\mathrm{T}\tilde{\mu}) = 0, &\quad \text{in } K \\ (\mathcal{A}\nabla\mathrm{T}\tilde{\mu}) \cdot \mathbf{n}^K = \tilde{\lambda}, &\quad \text{on } \partial K \end{aligned} \tag{17}$$

2. find $w = \tilde{T}f \in \tilde{\mathrm{H}}^1(\mathcal{T}_\mathcal{H})$ such that

$$\begin{aligned} \nabla \cdot (\mathcal{A}\nabla w) = f, &\quad \text{in } K \\ (\mathcal{A}\nabla w) \cdot \mathbf{n}^K = \frac{1}{|\partial K|}\int_K f\, d\mathbf{x}, &\quad \text{on } \partial K \end{aligned} \tag{18}$$

Note that the compatibility conditions for both (17) and (18) hold, and therefore the problems are well-posed. We can then write $\tilde{u} = T\tilde{\lambda} + \tilde{T}f$. Applying a static condensation in (16), we obtain the saddle-point problem of finding $(\tilde{\lambda}, \rho) \in \tilde{\Lambda} \times \mathrm{H}_0^{1/2}(\mathcal{E}_\mathcal{H})$ such that

$$\begin{array}{lcll} -\langle\tilde{\mu}, \mathrm{T}\tilde{\lambda}\rangle_{\mathcal{E}_\mathcal{H}} + \langle\tilde{\mu}, \rho\rangle_{\mathcal{E}_\mathcal{H}} & = & \langle\tilde{\mu}, \tilde{T}f\rangle_{\mathcal{E}_\mathcal{H}}, & \forall\, \tilde{\mu} \in \tilde{\Lambda}; \\ \langle\tilde{\lambda}, \xi\rangle_{\mathcal{E}_\mathcal{H}} & = & -\langle\lambda^0, \xi\rangle_{\mathcal{E}_\mathcal{H}}, & \forall\, \xi \in \mathrm{H}_0^{1/2}(\mathcal{E}_\mathcal{H}). \end{array} \tag{19}$$

We identify the first equation in (19) as the classical Dirichlet to Neumann map [1]

$$\mathrm{G} : \prod_{K \in \mathcal{T}_{\mathcal{H}}} H^{1/2}(\partial K) \longrightarrow \tilde{\Lambda}, \tag{20}$$

for which we associate each function $\rho \in \mathrm{H}_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ to its correspondent $\mathrm{G}\rho \in \tilde{\Lambda}$ defined as $\mathrm{G}\rho := \nabla(\mathrm{TG}\rho) \cdot \mathbf{n}^K$, where $\mathrm{TG}\rho$ is the weak solution of the local Dirichlet problem

$$\begin{aligned} \nabla \cdot (\nabla \mathrm{TG}\rho) = 0, &\quad \text{in } K, \\ \mathrm{TG}\rho = \rho, &\quad \text{on } \partial K, \end{aligned} \tag{21}$$

for all $K \in \mathcal{T}_{\mathcal{H}}$. Such variational problem is well-posed, since the bilinear form $\langle \cdot, \mathrm{T}\cdot \rangle_{\mathcal{E}_{\mathcal{H}}}$ is $\tilde{\Lambda}$-elliptic. For $\tilde{\mu} \in \tilde{\Lambda}$, we have

$$\begin{aligned} \langle \tilde{\mu}, \mathrm{T}\tilde{\mu} \rangle_{\mathcal{E}_{\mathcal{H}}} &= \sum_{K \in \mathcal{T}_{\mathcal{H}}} \langle \tilde{\mu}, \mathrm{T}\tilde{\mu} \rangle_{\partial K} = \sum_{K \in \mathcal{T}_{\mathcal{H}}} \int_K \mathcal{A} \nabla(\mathrm{T}\tilde{\mu}) \cdot \nabla(\mathrm{T}\tilde{\mu})\, dx \geq \sum_{K \in \mathcal{T}_{\mathcal{H}}} a_{\min} \|\nabla \mathrm{T}\tilde{\mu}\|_{0,K}^2 \\ &\geq \sum_{K \in \mathcal{T}_{\mathcal{H}}} C \|\mathrm{T}\tilde{\mu}\|_{1,K}^2 \geq \sum_{K \in \mathcal{T}_{\mathcal{H}}} C |\tilde{\mu}|_{-\frac{1}{2},\partial K}^2 \geq C |\tilde{\mu}|_{-\frac{1}{2},\mathcal{E}_{\mathcal{H}}}^2, \end{aligned} \tag{22}$$

where the last two inequalities follow from the Generalized Poincaré inequality and the injectivity of the operator T. Then, we replace $\tilde{\lambda} = \mathrm{G}(\rho - \tilde{\mathrm{T}} f)$ into (19) and apply an static condensation to get the global problem in terms of traces, that is, we seek for $\rho \in \mathrm{H}_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ such that

$$\langle \mathrm{G}\rho, \xi \rangle_{\mathcal{E}_{\mathcal{H}}} = -\langle \lambda^0, \xi \rangle_{\mathcal{E}_{\mathcal{H}}} + \langle \mathrm{G}\tilde{\mathrm{T}} f, \xi \rangle_{\mathcal{E}_{\mathcal{H}}}, \quad \forall\, \xi \in \mathrm{H}_0^{1/2}(\mathcal{E}_{\mathcal{H}}). \tag{23}$$

Well-posedness to (23) follows from the Theorem 1.1 below.

Theorem 1.1. *The bilinear form $\langle G\cdot, \cdot \rangle_{\mathcal{E}_{\mathcal{H}}} : H_0^{1/2}(\mathcal{E}_{\mathcal{H}}) \times H_0^{1/2}(\mathcal{E}_{\mathcal{H}}) \to \mathbb{R}$ is symmetric, bounded and coercive.*

The characterization of the exact solution is given by

$$u = u^0 + \mathrm{TG}\rho + (\mathrm{I} - \mathrm{TG})\tilde{\mathrm{T}} f. \tag{24}$$

2 Galerkin scheme

Let the finite dimensional subspaces $\Gamma_{\mathcal{H}_\Gamma} \subset \mathrm{H}_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ and $\tilde{\Lambda}_{\mathcal{H}_\Lambda} \subset \tilde{\Lambda}$. For simplicity, we assume that local problems have exact solutions, i.e., there are "no second level discretizations". Then, the stability of the method on the interfaces is conditioned to the compatibility condition between flows and traces: consider $\tilde{\Gamma}_{\mathcal{H}_\Gamma} := \Gamma_{\mathcal{H}_\Gamma} \cap \tilde{\mathrm{H}}^{1/2}(\mathcal{E}_{\mathcal{H}})$, where $\tilde{\mathrm{H}}_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ stands for space of "zero mean" functionals on the elements border, and define $\tilde{\Lambda}_{\mathcal{H}_0} \subset \tilde{\Lambda}$ such that

$$\tilde{\xi}_{\mathcal{H}_\Gamma} \in \tilde{\Gamma}_{\mathcal{H}_\Gamma} \quad \text{and} \quad \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\xi}_{\mathcal{H}_\Gamma} \rangle_{\partial K} = 0,\ \forall\, \tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_0, K},\ \forall\, K \in \mathcal{T}_{\mathcal{H}} \quad \Longrightarrow \quad \tilde{\xi}_{\mathcal{H}_\Gamma} = 0. \tag{25}$$

We illustrate an example of space functions that satisfy (25) depicted in the Figure 1. We assume that $\tilde{\Lambda}_{\mathcal{H}_\Lambda}$ is a finite dimensional space such that $\tilde{\Lambda}_{\mathcal{H}_0} \subset \tilde{\Lambda}_{\mathcal{H}_\Lambda} \subset \tilde{\Lambda}$. Finally, we define the operator $\mathrm{G}_h : \mathrm{H}_0^{1/2}(\mathcal{E}_{\mathcal{H}}) \to \tilde{\Lambda}_{\mathcal{H}_\Lambda}$, the discrete counterpart to G defined in (20). The Galerkin scheme of (23) consists to finding $\rho_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}$ such that

$$\langle \mathrm{G}_h \rho_{\mathcal{H}_\Gamma}, \xi_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}} = -\langle \lambda^0, \xi_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}} + \langle \mathrm{G}_h \tilde{\mathrm{T}} f, \xi_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}}, \quad \forall\, \xi_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}. \tag{26}$$

The following proposition is a stability result from the second compatibility condition (25).

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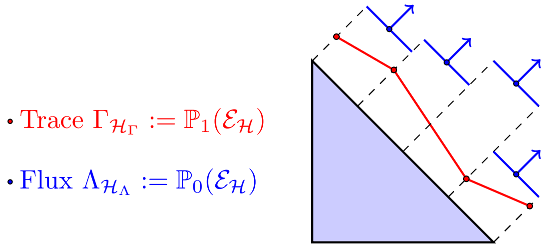

Figure 1: Compatible finite dimensional subspaces.

Proposition 2.1. *For $K \in \mathcal{T}_\mathcal{H}$, let $\tilde{\Lambda}_{\mathcal{H}_0} \subset \tilde{\Lambda}$ introduced in* (25)*. Then, there exists $\gamma_K > 0$ independent of $\mathcal{H}$ such that*

$$\sup_{\tilde{\mu}\in\tilde{\Lambda}} \frac{\langle \tilde{\mu}, \xi_{\mathcal{H}_\Gamma}\rangle_{\partial K}}{|\tilde{\mu}|_{-\frac{1}{2},\partial K}} \leq \gamma_K \sup_{\tilde{\mu}_{\mathcal{H}_\Lambda}\in\tilde{\Lambda}_{\mathcal{H}_0}} \frac{\langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \xi_{\mathcal{H}_\Gamma}\rangle_{\partial K}}{|\tilde{\mu}_{\mathcal{H}_\Lambda}|_{-\frac{1}{2},\partial K}}, \tag{27}$$

for all $\xi_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}$.

The following theorem ensures the well-posedness to (26).

Theorem 2.1. *The bilinear form $\langle G_h\cdot,\cdot\rangle_{\mathcal{E}_\mathcal{H}} : \Gamma_{\mathcal{H}_\Gamma} \times \Gamma_{\mathcal{H}_\Gamma} \to \mathbb{R}$ is symmetric and positive definite.*

The approximated solution is

$$u_h = u^0 + \mathrm{TG}_h\rho_{\mathcal{H}_\Gamma} + (\mathrm{I} - \mathrm{TG}_h)\tilde{\mathrm{T}}f; \tag{28}$$

Comparing to (24), we see that the only discretization involves G_h. That is because we assume exact solutions for the second level problems T and $\tilde{T}$.

3 Main result

We derive estimates for the approximations errors $\rho - \rho_{\mathcal{H}_\Gamma}$ and $\lambda - \lambda_{\mathcal{H}_\Lambda}$. The next result is based on the First Strang Lemma [3].

Theorem 3.1. *Let $(u, \lambda, \rho) \in H^1(\mathcal{T}_\mathcal{H}) \times \Lambda \times H_0^{1/2}(\mathcal{E}_\mathcal{H})$ be solution of the infinite dimensional variational formulation* (9) *and $(u_h, \lambda_{\mathcal{H}_\Lambda}, \rho_{\mathcal{H}_\Gamma}) \in V_h \times \Lambda_{\mathcal{H}_\Lambda} \times \Gamma_{\mathcal{H}_\Gamma}$ its approximated solution from Galerkin scheme* (28)*. Then, problem* (26) *is well-posed and*

$$\begin{aligned}
&|\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} \leq \inf_{\phi_{\mathcal{H}_\Gamma}\in\Gamma_\mathcal{H}} \left\{2|\rho - \phi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\phi_{\mathcal{H}_\Gamma})\right\} + E(\tilde{T}f);\\
&|\lambda - \lambda_{\mathcal{H}_\Lambda}|_\Lambda \leq |\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\rho_{\mathcal{H}_\Gamma}) + E(\tilde{T}f);\\
&|u - u_h|_{1,\mathcal{A},\mathcal{T}_\mathcal{H}} \leq |\lambda - \lambda_{\mathcal{H}_\Lambda}|_\Lambda
\end{aligned}$$

where, for $\phi \in H_0^{1/2}(\mathcal{E}_\mathcal{H})$,

$$E(\phi) := \inf_{\tilde{\mu}_{\mathcal{H}_\Lambda}\in\tilde{\Lambda}_{\mathcal{H}_\Lambda}} \left\{\left(C_T^{-1} + 1\right)|G\phi - \tilde{\mu}_{\mathcal{H}_\Lambda}|_\Lambda\right\};$$

Finally, the following weak continuity

$$\langle \mu_{\mathcal{H}_\Lambda}, u_h - \rho_{\mathcal{H}_\Gamma}\rangle_{\mathcal{E}_\mathcal{H}} = 0, \quad \forall\, \mu_{\mathcal{H}_\Lambda} \in \Lambda_{\mathcal{H}_\Lambda},$$

holds.

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4 The MH^{2}M method

We introduce the Multiscale Hybrid-Hybrid-Mixed method to get a numerical solution of (26). The infinite dimensional spaces are approximated by using polynomial space functions subject to the condition (25). Let the following pair of finite element spaces

$$\begin{aligned} \Gamma^1_{\mathcal{H}_\Gamma} &:= \left\{\xi_{\mathcal{H}_\Gamma} \in \mathrm{H}^{1/2}_0(\mathcal{E}_\mathcal{H});\ \xi_{\mathcal{H}_\Gamma}|_F \in \mathbb{P}_1(F),\ \forall\, F \in \mathcal{E}_\mathcal{H}\right\}; \\ \Lambda^0_{\mathcal{H}_\Lambda} &:= \prod_{K \in \mathcal{T}_\mathcal{H}} \left\{\mu_{\mathcal{H}_\Lambda} \in \mathrm{L}^2(\partial K);\ \mu_{\mathcal{H}_\Lambda}|_F \in \mathbb{P}_0(F),\ \forall\, F \in \partial K\right\}. \end{aligned} \tag{29}$$

They are illustrated in Figure 2. We obtain in the Theorem below the optimal convergence rates.

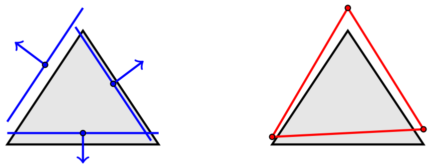

Figure 2: Representative functions of $\Lambda^0_{\mathcal{H}_\Lambda}$ and $\Gamma^1_{\mathcal{H}_\Gamma}$.

Theorem 4.1. *Under assumptions of Theorem 3.1, considering $u \in H^2(\mathcal{T}_\mathcal{H})$, let the finite element spaces $\Gamma^1_{\mathcal{H}_\Gamma}$ and $\Lambda^0_{\mathcal{H}_\Lambda}$ introduced in (29). Then, there exists constants $C > 0$ independent of $\mathcal{H}$ such that*

$$\begin{gathered} |\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} \leq C\mathcal{H}\|f\|_{0,\Omega}, \qquad |\lambda - \lambda_{\mathcal{H}_\Lambda}|_\Lambda \leq C\mathcal{H}\|f\|_{0,\Omega}; \\ |u - u_h|_{1,\mathcal{A},\mathcal{T}_\mathcal{H}} \leq C\mathcal{H}\|f\|_{0,\Omega}. \end{gathered} \tag{30}$$

5 Conclusion

The Multiscale Hybrid-Hybrid-Mixed method is derived from a hybrid-mixed three-field formulation and characterized by a symmetric elliptical problem. We relax the flux defining it on each element boundary, so that different flux meshes can be taken for each element. Although there is no conformity to the numerical solution, the trace and flux are conform, that is, our method preserves mass conservation for the flux. Continuous and discrete inf-sup conditions hold. We introduce a pair of compatible finite element spaces and error estimates.

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