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On the family of polynomials generated by a four-term recurrence relation

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Abstract. This paper investigates the distribution, simplicity, and monotonicity of the zeros of even and odd polynomials generated by a four-term recurrence relation and linear coefficients.

Palavras-chave. Even and odd polynomials, Four-term recurrence relation, Zeros.

1 Introduction

Sequences of polynomials generated by recurrence relations have been explored since the 18th century. A classical example is the three-term recurrence relation related to the real orthogonal polynomial sequence [2, 4]. However, not much is known in the case of polynomials generated by a four-term recurrence relation. We can cite the references [1, 5, 6] as an example of recent studies involving these classes of polynomials.

In this paper, we consider the polynomial sequence satisfying

$$Q_n(z) = zQ_{n-1}(z) - b_0 Q_{n-2}(z) - c_1 z Q_{n-3}(z), \tag{1}$$

with $b_0, c_1 \in \mathbb{R} - \{0\}$, $Q_0(z) = 1$ and $Q_{-n}(z) = 0$, for all $n \in \mathbb{N}$.

This case was studied in reference [1], where the author presented necessary and sufficient conditions on the coefficients b_0 and c_1 such that all the zeros of the polynomial $Q_n(z)$ are real, for all n. The main result of [1] is the following.

Theorem 1.1. *The zeros of $Q_n(z)$ are real if and only if $b_0 > 0$ and $-\frac{c_1}{b_0} := \alpha \leq \frac{1}{9}$, in which case they lie on the interval $I := \sqrt{b_0}(-\lambda, \lambda)$, where*

$$\lambda := \frac{4}{\left(\frac{3\alpha+1+\sqrt{9\alpha^2-10\alpha+1}}{-5\alpha+1+\sqrt{9\alpha^2-10\alpha+1}}\right)^{3/2}\left(-5\alpha+1+\sqrt{9\alpha^2-10\alpha+1}\right)}.$$

Futhermore, if $\mathcal{Z}(Q_n)$ is the set of zeros of $Q_n(z)$, then $\bigcup_{n=0}^{\infty} \mathcal{Z}(Q_n(z))$ is dense on I.

Observe that, if $b_0 > 0$ and $c_1 > 0$, from this result it follows that all the zeros of $Q_n(z)$, represented by $z_{1n}, z_{2,n}, \ldots, z_{n,n}$, are real. This observation will be used in Theorem 2.1, which is the main result of this paper.

In the following, we present some properties of polynomial $Q_n(z)$, that will be used to analyze the distribution of the zeros according to the signal of b_0 and c_1. We may see a complete study of this topic in [3].

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2 Properties of $Q_n(z)$

We can find proof of these lemmas in [3].

Lemma 2.1. *If n is even (odd), the polynomial $Q_n(z)$ is even (odd).*

Lemma 2.2. *If n is odd, $z = 0$ is zero of $Q_n(z)$. Furthermore, if $\operatorname{sgn}(b_0) = \operatorname{sgn}(c_1)$, $z = 0$ is a simple zero. For n even, $Q_n(0) \neq 0$.*

Lemma 2.3. *For any n, $n > 1$, if $\operatorname{sgn}(b_0) = \operatorname{sgn}(c_1)$, then the three consecutive polynomials Q_n, Q_{n-1} and Q_{n-2} do not have common zeros.*

As we mentioned before, if $b_0 > 0$ and $c_1 > 0$ in (1), from Theorem 1.1, it follows that all the zeros of $Q_n(z)$ are real. In the following result we prove that under this condition, the zeros of $Q_n(z)$ are distinct and satisfy the interlacing property.

From Lemma 2.1, it follows that the zeros of $Q_n(z)$ are symmetric with respect to the origin. So, $Q_n(z)$ has $\lfloor n/2 \rfloor$ positive zeros, denoted by $z_{k,n}$, $k = 1, \ldots, \lfloor n/2 \rfloor$, and $\lfloor n/2 \rfloor$ negative zeros, represented by $z_{k,n}$, $k = n/2 + 1, \ldots, n$ (for n even) and $k = \lfloor n/2 \rfloor + 2, \ldots, n$ (for n odd). If n is odd, $z_{\lfloor n/2 \rfloor + 1, n} = 0$.

Theorem 2.1. *If $b_0 > 0$ and $c_1 > 0$ in (1), $Q_n(z)$ has n distinct and real zeros $z_{1,n} > z_{2,n} > \ldots > z_{n,n}$ such that*

- *if n is even,*

$$\begin{aligned} &z_{1,n} > z_{1,n-1} > z_{1,n-2} > z_{2,n} > z_{2,n-1} > z_{2,n-2} > \ldots > z_{n/2,n} \\ &> z_{n/2,n-1} = 0 > z_{n/2+1,n} > z_{n/2,n-2} > z_{n/2+1,n-1} > z_{n/2+2,n} \\ &> \ldots > z_{n-3,n-2} > z_{n-2,n-1} > z_{n-1,n} > z_{n-2,n-2} > z_{n-1,n-1} > z_{n,n}; \end{aligned} \tag{2}$$

- *if n is odd,*

$$\begin{aligned} &z_{1,n} > z_{1,n-1} > z_{1,n-2} > z_{2,n} > z_{2,n-1} > z_{2,n-2} > \ldots > z_{\lfloor n/2 \rfloor,n} \\ &> z_{\lfloor n/2 \rfloor,n-1} > z_{\lfloor n/2 \rfloor,n-2} = 0 = z_{\lfloor n/2 \rfloor+1,n} > z_{\lfloor n/2 \rfloor+1,n-1} \\ &> z_{\lfloor n/2 \rfloor+2,n} > \ldots > z_{n-3,n-2} > z_{n-2,n-1} > z_{n-1,n} \\ &> z_{n-2,n-2} > z_{n-1,n-1} > z_{n,n}, \end{aligned} \tag{3}$$

where $z_{1,n-1}, z_{2,n-1}, \ldots, z_{n-1,n-1}$ and $z_{1,n-2}, z_{2,n-2}, \ldots, z_{n-2,n-2}$ are the zeros of $Q_{n-1}(z)$ and $Q_{n-2}(z)$, respectively.

Proof. Firstly, we will analyse the behaviour of the zeros of Q_1, Q_2, Q_3 and Q_4:

1. $Q_1(z) = z$ and then $z_{1,1} = 0$.
2. $Q_2(z) = z^2 - b_0$, $z_{1,2} = \sqrt{b_0}$ and $z_{2,2} = -\sqrt{b_0}$. Consequently,

$$z_{1,2} > z_{1,1} > z_{2,2}.$$

3. $Q_3(z) = zQ_2(z) - (b_0 + c_1)z = z[(z - z_{1,2})(z - z_{2,2}) - (b_0 + c_1)]$. Observe that $Q_3(0) = 0$, $Q_3(z_{1,2}) < 0$ and $Q_3(z_{2,2}) > 0$. So, $z_{1,3} \in (z_{1,2}, \infty)$ and $z_{3,3} \in (-\infty, z_{2,2})$. Then,

$$z_{1,3} > z_{1,2} > z_{1,1} = 0 = z_{2,3} > z_{2,2} > z_{3,3}.$$

4. $Q_4(z) = z[(z-z_{1,3})(z-z_{2,3})(z-z_{3,3})-c_1Q_1(z)]-b_0(z-z_{1,2})(z-z_{2,2})$. Note that $Q_4(z_{3,3}) < 0$, $Q_4(z_{2,3}) > 0$ and $Q_4(z_{1,3}) < 0$. So, $z_{1,4} \in (z_{1,3}, \infty)$, $z_{2,4} \in (z_{2,3}, z_{1,3})$, $z_{3,4} \in (z_{2,3}, z_{z_{3,3}})$ and $z_{4,4} \in (-\infty, z_{3,3})$. Furthermore, $Q_4(z_{1,2}) < 0$ and $Q_4(z_{2,2}) < 0$. Consequently,
$$z_{1,4} > z_{1,3} > z_{1,2} > z_{2,4} > z_{2,3} = 0 > z_{3,4} > z_{2,2} > z_{3,3} > z_{4,4}.$$

By induction hypothesis, we assume that for some $n \geq 3$, the zeros of the polynomials Q_n, Q_{n-1} and Q_{n-2} satisfy the relations (2), for n even, and (3), for n odd.

We will prove that those inequalities work for the zeros of Q_{n+1}, Q_n and Q_{n-1}. Firstly, note that
$$Q_{n+1}(z) = z\prod_{j=1}^{n}(z-z_{j,n}) - b_0\prod_{j=1}^{n-1}(z-z_{j,n-1}) - c_1 z\prod_{j=1}^{n-2}(z-z_{j,n-2}). \tag{4}$$

If $n+1$ is even, from (4) it follows that
$$\begin{aligned} \operatorname{sgn}(Q_{n+1}(z_{k,n})) &= (-1)^k,\ k=1,\ldots,n,\\ \operatorname{sgn}(Q_{n+1}(z_{k,n-1})) &= \begin{cases} (-1)^k, & k=1,\ldots,\frac{n-1}{2},\\ (-1)^{k+1}, & k=\frac{n+1}{2},\ldots,n-1 \end{cases}. \end{aligned}$$

Furthermore, $\lim_{z\to\infty} Q_{n+1}(z) > 0$ and $\lim_{z\to-\infty} Q_{n+1}(z) > 0$.

With this, we have the existence of $n+1$ real zeros of $Q_{n+1}(z)$ such that
$$\begin{aligned} &z_{1,n+1} > z_{1,n} > z_{1,n-1} > z_{2,n+1} > z_{2,n} > z_{2,n-1} > \ldots > z_{(n+1)/2,n+1}\\ &> z_{(n+1)/2,n} = 0 > z_{(n+3)/2,n+1} > z_{(n+1)/2,n-1} > z_{(n+3)/2,n} > z_{(n+5)/2,n+1}\\ &> \ldots > z_{n-2,n-1} > z_{n-1,n} > z_{n,n+1} > z_{n-1,n-1} > z_{n,n} > z_{n+1,n+1}, \end{aligned}$$

from which we complete the proof for $n+1$ even.

If $n+1$ is odd, from (4) it follows that
$$\begin{aligned} \operatorname{sgn}(Q_{n+1}(z_{k,n})) &= (-1)^k,\ k=1,\ldots,n,\\ \operatorname{sgn}(Q_{n+1}(z_{k,n-1})) &= \begin{cases} (-1)^k, & k=1,\ldots,\lfloor (n+1)/2\rfloor-1,\\ (-1)^{k+1}, & k=\lfloor (n+1)/2\rfloor+1,\ldots,n-1 \end{cases},\\ Q_{n+1}(z_{\lfloor (n+1)/2\rfloor,n-1}) &= 0. \end{aligned}$$

Furthermore, $\lim_{z\to\infty} Q_{n+1}(z) > 0$ and $\lim_{z\to-\infty} Q_{n+1}(z) < 0$.

With this, we have the existence of $n+1$ real zeros of $Q_{n+1}(z)$ such that
$$\begin{aligned} &z_{1,n+1} > z_{1,n} > z_{1,n-1} > z_{2,n+1} > z_{2,n} > z_{2,n-1} > \ldots\\ &> z_{\lfloor (n+1)/2\rfloor,n} > z_{\lfloor (n+1)/2\rfloor+1,n} > z_{\lfloor (n+1)/2\rfloor,n-1} = 0 = z_{\lfloor (n+1)/2\rfloor+1,n+1}\\ &> z_{\lfloor (n+1)/2\rfloor+2,n+1} > \ldots > z_{n-2,n-1} > z_{n-1,n} > z_{n,n+1}\\ &> z_{n-1,n-1} > z_{n,n} > z_{n+1,n+1}, \end{aligned}$$

from which the proof for an odd $n+1$ follows. □

If $b_0 > 0$, $c_1 < 0$ and $-\dfrac{c_1}{b_0} \leq \dfrac{1}{9}$ in (1), from Theorem 1.1 it follows that all the zeros of $Q_n(z)$ are real. Experiments show that the interlacing property mentioned in Theorem 2.1 is valid. So, we proposed the following conjecture:

Conjecture 2.1. *If $b_0 > 0$, $c_1 < 0$ and $-\dfrac{c_1}{b_0} \leq \dfrac{1}{9}$ in* (1)*, $Q_n(z)$ has n distinct and real zeros $z_{1,n} > z_{2,n} > \ldots > z_{n,n}$ such that the relations* (2) *and* (3) *are satisfied.*

3 Numerical example

To exemplify the properties presented in the previous section, we shall consider $b_0 = 0.01$ and $c_1 = 1$. So, we have

$$Q_n(z) = zQ_{n-1}(z) - 0,01Q_{n-2}(z) - zQ_{n-3}(z), \tag{5}$$

with $Q_0(z) = 1$ and $Q_{-n}(z) = 0$, for all $n \in \mathbb{N}$. From the equation (5), it follows that, for $n = 5,\ 6$ and 7,

$$\begin{aligned} Q_5(z) &= z^5 - \frac{76}{25}z^3 + \frac{203}{10000}z, \\ Q_6(z) &= z^6 - \frac{81}{20}z^4 + \frac{5303}{5000}z^2 - 10^{-6}, \\ Q_7(z) &= z^7 - \frac{253}{50}z^5 + \frac{3121}{1000}z^3 - \frac{19}{62500}z. \end{aligned}$$

The next figure display the zeros of $Q_5(z),\ Q_6(z)$ and $Q_7(z)$, represented by green, blue and red points, respectively. Counting the zeros of each polynomial in Figure 1 on the left, we can miss three zeros from $Q_7(z)$, two zeros from $Q_6(z)$, and one zero from $Q_5(z)$. This bad impression is clarified by enlarging the figure close enough to the origin, as in the figure on the right. The zeros that we cannot see clearly, due to the scale of Figure 1 on the left, are

$$\begin{aligned} z_{3,7} \approx 0,0099,\ z_{4,7} = 0 \text{ e } z_{5,7} \approx -0,0099,\ \text{zeros of } Q_7(z), \\ z_{3,6} \approx 0,001 \text{ e } z_{4,6} \approx -0,001,\ \text{zeros of } Q_6(z), \\ z_{3,5} = 0,\ \text{zero of } Q_5(z). \end{aligned}$$

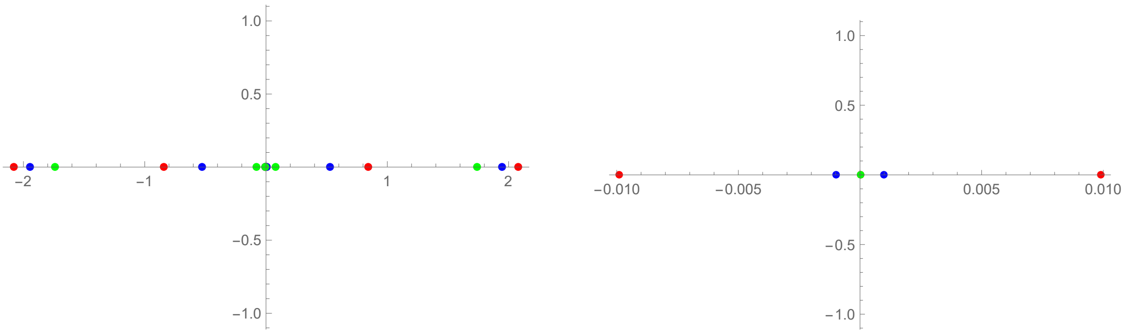

Figure 1: Representation of zero interlacing property of $Q_5(z),\ Q_6(z)$ and $Q_7(z)$.

Note that the origin is a simple zero of $Q_5(z)$ and $Q_7(z)$, as mentioned in Lemma 2.2. Also, it is easy to see that they do not have common zeros. Furthermore, as predicted in Theorem 2.1, the zeros satisfy the interlacing property.

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