Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 10, n. 1, 2023.

Trabalho apresentado no XLII CNMAC, Universidade Federal de Mato Grosso do Sul - Bonito - MS, 2023

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Chaotic Behavior in Diffusively Coupled Systems

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Abstract. We study emergent oscillatory behavior in networks of diffusively coupled nonlinear ordinary differential equations. Starting from a situation where each isolated node possesses a globally attracting equilibrium point, we give, for an arbitrary network configuration, general conditions for the existence of the diffusive coupling of a homogeneous strength which makes the network dynamics chaotic. The method is based on the theory of local bifurcations we develop for diffusively coupled networks. We, in particular, introduce the class of the so-called versatile network configurations and prove that the Taylor coefficients of the reduction to the center manifold for any versatile network can take any given value.

Key-words. Versatile Networks, Chaos, Diffusive Coupling, Dynamical Systems.

1 Introduction

Coupled dynamical systems play a prominent role in biology [6], chemistry [9], physics and other fields of science [18]. Understanding the emergent dynamics of such systems is a challenging problem, depending starkly on the underlying interaction structure [7, 10, 11, 13, 16].

In the early fifties, Turing thought of the emergent oscillatory behavior due to diffusive interaction as a model for morphogenesis [20]. We note that weak coupling of globally stable individual systems cannot alter the stability of the homogeneous regime, this is the globally attracting state. At the same time, no matter what the individual dynamics are, the strong diffusive coupling by itself stabilizes the homogeneous regime. Therefore, the idea that the intermediate strength diffusive coupling can create a non-trivial collective behavior is quite paradoxical. However, in the mid-seventies, Smale [17] proposed an example of diffusion-driven oscillations. He considered two 4th-order diffusively coupled differential equations, which by themselves have globally asymptotically stable equilibrium points. Once the diffusive interaction is strong enough, the coupled system exhibits oscillatory behavior. Smale posed a problem of finding conditions under which diffusively coupled systems would oscillate.

Tomberg and Yakubovich [19] proposed a solution to this problem for the diffusive interaction of two systems with scalar nonlinearity. For networks, Pogromsky, Glad, and Nijmeijer [15] showed that diffusion-driven oscillations can result from an Andronov-Hopf bifurcation. Moreover, they presented conditions to ensure the emergence of oscillations for general graphs. While this provides a good picture of the instability leading to periodic oscillations, there is evidence that the diffusive coupling may also lead to chaotic oscillations. Indeed, Kocarev and Janic [8] provided numerical evidence that two isolated Chua circuits having globally stable fixed points may exhibit chaotic

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behavior when diffusively coupled. Along the same lines, Perlikowski and co-authors [14] investigated numerically the dynamics of rings of unidirectionally coupled Duffing oscillators. Starting from the situation where each oscillator has an exponentially stable equilibrium point, once the oscillators are coupled akin to diffusion the authors found a great variety of phenomena such as rotating waves, the birth of periodic dynamics, as well as chaotic dynamics.

Drubi, Ibanez, and Rodriguez [3] studied two diffusively coupled Brusselators. Starting from a situation where the isolated systems have a globally stable fixed point, they proved that the unfolding of the diffusively coupled system can display a homoclinic loop with an invariant set of positive entropy.

We provide general conditions for diffusively coupled identical systems to exhibit chaotic oscillations. We describe necessary and sufficient conditions (the so-called skewness condition) on the linearization matrix at an exponentially stable equilibrium point of the isolated system such that for any network of such systems there exists a diffusive coupling matrix such that the network has a nilpotent singularity and thus a nontrivial center manifold. When the network structure satisfies an extra condition, which we call versatility, we show that Taylor coefficients of the vector field on the center manifold are in general position. This allows us to employ the theory of bifurcations of nilpotent singularities due to Arneodo, Coullet, Spiegel and Tresser [2] and Ibanez and Rodríguez [5] and to show that when the isolated system is at least four-dimensional, invariant sets of positive entropy (i.e., chaos) emerge in such networks.

2 The model

We consider ordinary differential equations $\dot{x} = f(x)$ with $f \in \mathcal{C}^\infty(U, \mathbb{R}^n)$, $n \in \mathbb{N}$ for some open set $U \subset \mathbb{R}^n$. We assume that f has an exponentially stable fixed point in U; with no loss of generality, we put the origin of coordinates to this point. We study a network of such systems coupled together according to a given graph structure by means of a diffusive interaction. Namely, we consider the following equation:

$$\dot{x}_i = f(x_i) + \alpha \sum_{j=1}^{N} w_{ij} D(x_j - x_i), \quad i = 1, \ldots, N, \tag{1}$$

where $\alpha > 0$ is the coupling strength, $W = (w_{ij})$ is the adjacency matrix of the graph, thus, $w_{ij} = 1$ if nodes i and j are connected and zero otherwise. Moreover, D is a positive-definite matrix (that is, $x^T Dx > 0$ for all non-zero vectors x).

The homogeneous regime $x = 0$ persists for every value of the coupling strength α. It keeps its stability at small α and is, typically, stable at sufficiently large α. However, at intermediate values of the coupling strength, the stability of the homogeneous regime can be lost. Our goal is to investigate the accompanying bifurcations. The difficulty is that the structure of system (1) is quite rigid: all network nodes are the same (are described by the same function f) and the diffusion coupling αD is the same for any pair of nodes. Therefore, the genericity arguments, standard for the bifurcation theory, cannot be readily applied and must be re-examined.

2.1 Informal statement of main results

Our main goal is to give conditions for the emergence of non-periodic dynamics in the system (1). Denote by $A = Df(0)$ the linearization matrix $n \times n$ of the individual uncoupled system at zero. Recall that matrix A is Hurwitz when all its eigenvalues have strictly negative real parts. Our main result can be stated as follows.

Suppose that, for some orthogonal basis, the Hurwitz matrix A has m positive entries on the diagonal. Then, there exists a positive-definite matrix D such that the linearization of system (1) *at the homogeneous equilibrium at zero has a zero eigenvalue of multiplicity, at least m for a certain value of the coupling parameter $\alpha > 0$. If the network satisfies a condition we call versatility, for an appropriate choice of the nonlinearity of f, the corresponding center manifold has dimension precisely m and the Taylor coefficients of the restriction of the system on the center manifold can take on any prescribed value.*

The last statement means that the bifurcations of the homogeneous state of a versatile network follow the same scenarios as general dynamical systems. Applying the results for triple instability [3, 5] we obtain the following result.

For $n \geqslant 4$, for any generic 2-parameter family of nonlinearities f and any versatile network graph, one can find the positive-definite matrix D such that the homogeneous state of the coupled system (1) *has a triple instability at a certain value of the coupling strength α, leading to chaotic dynamics for a certain region of parameter values.*

The condition on the Jacobian of the isolated dynamics can be understood in a geometric sense as follows. We write $\dot{x} = f(x) = Ax + \mathcal{O}(|x|^2)$. We claim that if a nonzero vector $x_0 \in \mathbb{R}^n$ exists for which $\langle x_0, Ax_0\rangle > 0$, then there are points arbitrarily close to the origin, whose forward orbit has its Euclidean distance to the origin increasing for some time, before coming closer to the (stable) origin again. To see why, consider $\|x(t)\|^2 = \langle x(t), x(t)\rangle$, then it follows that $\frac{d}{dt}\|x\|^2 = 2\langle Ax, x\rangle + \mathcal{O}(|x|^3)$, so $\langle Ax, x\rangle > 0$ implies the growth of this derivative.

The property of versatility holds for graphs with heterogeneous degrees – the simplest example is a star network. In a sense, versatility means that the network is not very symmetric. Given a graph, one verifies whether the versatility property holds by evaluating the eigenvectors of the graph's Laplacian matrix, so it is an effectively verifiable property.

3 Main results

We start by introducing the basic concepts involved in the setup of the problem.

3.1 Graphs

A graph G is an ordered pair (V, E), where V is a non-empty set of vertices and E is a set of edges connecting the vertices. We assume both to be finite and the graph to be undirected. The order of the graph G is $|V| = N$, its number of vertices, and the size is $|E|$, its number of edges. We will not consider graphs with self-loops. The degree of a vertex is the number of edges that are connected to it.

$$k_i = \sum_{j=1}^{N} w_{ij}, \tag{2}$$

for, $i = 1, \ldots, N$. We also define $K = \text{diag}\{k_1, \ldots, k_N\}$ to be the diagonal matrix of vertex degrees.

We only consider undirected graphs G, meaning that a vertex i is connected with a vertex j if and only if it is *vice versa*. Thus, the adjacency matrix W is a symmetric matrix. In this context, there is another important matrix related to the graph G, which is the well-known *Laplacian* discrete matrix L_G. It is defined by:

$$L_G = K - W,$$

so that each entry l_{ij} of L_G can be written as

$$l_{ij} = \delta_{ij} k_i - w_{ij}, \quad i, j = 1, \dots, N, \tag{3}$$

where δ_{ij} is *Kronecker's* delta. The matrix L_G provides us with important information about connectivity and synchronization of the network. It follows from Gershgorin disk theorem [4] that L_G is positive semi-definite and thus its eigenvalues can be ordered as

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1} \leq \lambda_N,$$

and let $\{v_1, \dots, v_N\}$ be the corresponding eigenvectors. We assume the network is connected. This implies that the eigenvalue $\lambda_1 = 0$ is simple.

We are interested in a well-behaved class of graphs G whose structure induces a special property of the associated Laplacian matrix. This property will be the existence of an eigenvector where the sum of certain coordinate powers is non-vanishing, which corresponds to a simple eigenvalue of L_G. To this end, we define:

Definition 3.1 (ρ-versatile graphs)**.** *Let $G = (V, E)$ be a graph and $\rho \in \mathbb{N}$ a positive integer. We say that G is ρ-versatile for the eigenvalue-eigenvector pair (λ, v) with $\lambda > 0$, if the Laplacian matrix L_G has a simple eigenvalue λ with corresponding eigenvector $v = (\nu_1, \dots, \nu_N)$, satisfying*

$$\sum_{i=1}^{N} \nu_i^{\ell} \neq 0, \quad \forall \ell = 2, \dots, \rho + 1. \tag{4}$$

Note that any eigenvector $v = (\nu_1, \dots, \nu_N)$ for a non-zero eigenvalue necessarily satisfies $\sum_{i=1}^{N} \nu_i = 0$. This is because ν is orthogonal to the eigenvector $(1, \dots, 1)$ for the eigenvalue 0.

3.2 Parametrization

We show that a system of diffusively coupled stable systems can display a wide variety of dynamic behavior, including the onset of chaos. As the coupling strength α increases, a non-trivial center manifold can emerge with no general restrictions on the Taylor coefficients of the reduced dynamics.

Note that we may alternatively write Equation (1) in terms of the Laplacian:

$$\dot{x}_i = f(x_i) - \alpha \sum_{j=1}^{N} l_{ij} D x_j, \quad i = 1, \dots, N. \tag{5}$$

Let $X := col(x_1, \dots, x_N)$ denote the vector formed by stacking x_i's in a single column vector. In the same way, we define $F(X) := col(f(x_1), \dots, f(x_N))$. We obtain the compact form for equations (1) and (5) given by

$$\dot{X} = F(X) - \alpha (L_G \otimes D) X, \tag{6}$$

where $\otimes$ stands for the Kronecker product. In order to analyze systems of the form (1), we allow f depending on a parameter ε taking values in some open neighborhood of the origin $\Omega \subseteq \mathbb{R}^d$. For simplicity, we assume the fixed point at the origin persists:

$$f(0; \varepsilon) = 0 \qquad \forall \, \varepsilon \in \Omega.$$

We assume the origin to be exponentially stable for $\varepsilon = 0$, from which stability follows for sufficiently small ε as well. Note that the non-linear diagonal map F now depends on the parameter ε as well.

We start with our working definition of center manifold reduction.

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Definition 3.2. *Let*

$$H : \mathbb{R}^n \times \Omega \to \mathbb{R}^n \tag{7}$$

be a family of vector fields on $\mathbb{R}^n$, parameterized by a variable, ε in an open neighborhood of the origin $\Omega \subseteq \mathbb{R}^d$. Assume that $H(0;\varepsilon) = 0$ for all $\varepsilon \in \Omega$, and denote by $\mathcal{E}^c \subseteq \mathbb{R}^n$ the center subspace of the Jacobian $D_x H(0;0)$ in the direction of $\mathbb{R}^n$. A (local) ***parameterized center manifold*** *of the system* (7) *is a (local) center manifold of the unparameterized system $\tilde{H}$ on $\mathbb{R}^n \times \Omega$, given by*

$$\tilde{H}(x;\varepsilon) = (H(x;\varepsilon), 0) \in \mathbb{R}^n \times \mathbb{R}^d, \tag{8}$$

for $x \in \mathbb{R}^n$ and $\varepsilon \in \Omega$. We say that the parameterized center manifold is of dimension $\dim(\mathcal{E}^c)$*, and is parameterized by d variables. Under the assumptions on H, the center subspace of $\tilde{H}$ at the origin is equal to $\mathcal{E}^c \times \mathbb{R}^d$. We can show that the dynamics on the center manifold of Equation* (8) *is conjugate to that of a locally defined system*

$$\tilde{R}(x_c;\varepsilon) = (R(x_c;\varepsilon), 0), \tag{9}$$

on $\mathcal{E}^c \times \Omega$, where the conjugation respects the constant-ε fibers. The map R satisfies $R(0;\varepsilon) = 0$ for all ε for which this local expression is defined, and we have $D_{x_c} R(0;0) = D_x H(0;0)|_{\mathcal{E}^c} : \mathcal{E}^c \to \mathcal{E}^c$. We will refer to $R : \mathcal{E}^c \times \Omega \to \mathcal{E}^c$ as a ***parameterized reduced vector field*** *of H.*

In the definition above, the constant and linear terms of the parameterized reduced vector field R are given. Motivated by this, we will write $H^{[2,\rho]}$ for any map H to denote the non-constant, non-linear terms in the Taylor expansion around the origin of H, up to terms of order ρ. In other words, we have

$$H(x) = H(0) + DH(0)x + H^{[2,\rho]}(x) + \mathcal{O}(||x||^{\rho+1}).$$

Given vector spaces W and W', we will use $\mathcal{P}_2^l(W;W')$ to denote the linear space of polynomial maps from W to W' with terms of degree 2 through l. It follows that $H^{[2,l]} \in \mathcal{P}_2^l(W;W')$ for $H : W \to W'$.

We are interested in the situation where the domain of H involves some parameter space Ω, in which case $H^{[2,\rho]}$ involves all non-constant, non-linear terms up to order ρ in both types of variables (parameter and phase space). For instance, if H is a map from $\mathbb{R} \times \Omega$ to $\mathbb{R}$ with $\Omega \subseteq \mathbb{R}$, then $H^{[2,3]}(x;\varepsilon)$ involves the terms

$$a_1x^2, a_2x\varepsilon, a_3\varepsilon^2, a_4x^3, a_5x^2\varepsilon, a_6x\varepsilon^2 \text{ and } a_7\varepsilon^3,$$

with some constants a_i. Note that a condition on H might put restraints on $H^{[2,\rho]}$ as well. For instance, if $H(0;\varepsilon) = 0$ for all $\varepsilon \in \Omega$, then $H^{[2,3]}(x;\varepsilon)$ does not involve the terms ε^2 and ε^3.

3.3 Main theorems

We now formulate the main theorem, along with an important corollary.

Theorem 3.1 (Main Theorem)**.** *For any $\alpha \geq 0$, consider the ε-family of network dynamical systems given by*

$$\dot{X} = F(X;\varepsilon) - \alpha(L_G \otimes D)X. \tag{10}$$

Denote by $A = D_x f(0;0)$ the Jacobian of the isolated dynamics. If there exist m mutually orthogonal vectors $x_1, \dots, x_m$ such that $\langle x_i, Ax_i \rangle > 0$, then there exists a positive-definite matrix D together with a number $\alpha^* > 0$ such that the system of Equation (10) *has a local parameterized center manifold of dimension at least m for $\alpha = \alpha^*$.*

Suppose that the graph G is ρ-versatile for the pair (λ, v). After an arbitrarily small perturbation to A if needed, there exists a positive-definite matrix D and a number $\alpha^* > 0$ such that the following holds:

1. *The system of Equation* (10) *has a local parameterized center manifold of dimension exactly* m *for* $\alpha = \alpha^*$.
2. *Denote by* $R : \mathcal{E}^c \times \Omega \to \mathcal{E}^c$ *the corresponding parameterized reduced vector field, then* $R(0;\varepsilon) = 0$ *for all* $\varepsilon \in \Omega$ *and* $D_x R(0;0) : \mathcal{E}^c \to \mathcal{E}^c$ *is nilpotent.*
3. *The higher order terms* $R^{[2,\rho]}$ *can take on any value in* $\mathcal{P}_2^{\rho}(\mathcal{E}^c \times \Omega; \mathcal{E}^c)$ *(subject to* $R^{[2,\rho]}(0;\varepsilon) = 0$*) as* $f^{[2,\rho]}$ *is varied (subject to* $f^{[2,\rho]}(0;\varepsilon) = 0$*).*

The above result guarantees the existence of the center manifold and the reduced vector field. When the dimension of the isolated dynamics is at least 4, the reduced vector field can exhibit invariant sets of positive entropy, as the following result shows.

Corollary 3.1 (Chaos)**.** *Assume the conditions of Theorem* (3.1) *to hold for* $m = 3$ *and* $\rho = 2$. *Then, in a generic* 3*-parameter system, we have the emergence of chaos through the formation of a Shilnikov loop on the center manifold. In particular, chaos can form this way in a system of* 4*-dimensional nodes coupled diffusively in a network.*

4 Final considerations

It is possible that a similar theory can be developed for the Andronov-Hopf bifurcation in diffusively coupled networks (an analysis of diffusion-driven Andronov-Hopf bifurcations was undertaken in [15] but the question of genericity of the restriction of the network system to the central manifold was not addressed there).

We also point out that symmetry is often instrumental in explaining and predicting anomalous behavior in network dynamical systems [1, 11, 12].

The network of just two symmetrically coupled systems has the corresponding graph Laplacian that is not versatile, yet the emergence of chaos via the triple instability has been established in [3] for the system of two diffusively coupled Brusselators. In general, we do not know when the genericity of the Taylor coefficients of the center-manifold reduced vector field would hold if the graph is not versatile or when graph symmetries would impose conditions on the dynamics that forbid the existence of limiting sets of positive entropy.

Acknowledgments

We thank Edmilson Roque and Jeroen Lamb for enlightening discussions. TP was supported in part by FAPESP Cemeai Grant No. 2013/07375-0 and is a Newton Advanced Fellow of the Royal Society NAF\R1\180236. EN was supported by FAPESP grant 2020/01100-2. TP and EN were partially supported by Serrapilheira Institute (Grant No. Serra-1709-16124). FCQ was supported by CAPES. DT was supported by Leverhulme Trust grant RPG-2021-072.

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