Trabalho apresentado no XLII CNMAC, Universidade Federal de Mato Grosso do Sul - Bonito - MS, 2023

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Probability Models Generated via Line Integral and Joint Life Insurance Application

Nikolai Kolev[1]
IME-USP, São Paulo, SP

Abstract. We construct via line integral and characterize a class of bivariate continuous distributions with a multiplicative representation of the sum of hazard gradient components. The corresponding joint survival function is a solution of functional equation allowing to generate new members of the class. We apply a particular member to fit a big Canadian joint life insurance data set improving the inference and conclusions made by of another authors.

Key words. Characterization, Functional equation, Hazard gradient vector, Line integral

1 Introduction and preliminaries

Many continuous multivariate distributions have been developed to model the dependence structure of data sets in a wide range of areas. In general, the goal is to construct a multivariate distribution which fits properly a particular data with specific features. In general, the building procedure has to take into account that the observed data are output of a real process following some physical law. The functional form of the sum of gradient vector components plays a crucial role to describe the nature of a process under interest. This is our motivation to use the line integral of gradient vector representation of the joint survival function, firstly introduced in probability theory and its applications in [5]. The aim is to construct a family of multivariate probability distributions with pre-specified properties.

For simplicity, we will examine the bivariate case. More specifically, let us consider a non-negative bivariate continuous random vector (X, Y) defined by its joint survival function $S(x, y) = P(X > x, Y > y)$ for all $x, y \geq 0$. If the first partial derivatives of $S(x, y)$ exist, the quantities $r_1(x, y) = \frac{\partial}{\partial x}\Big[- \ln S(x, y)\Big]$ and $r_2(x, y) = \frac{\partial}{\partial y}\Big[- \ln S(x, y)\Big]$ can be interpreted as the univariate failure rates of conditional distributions of each variate, given certain inequality of the remainder, i.e., of $(X \,|\, Y > y)$ and $(Y \,|\, X > x)$. Observe that $r_1(x, 0) = r_X(x)$ and $r_2(0, y) = r_Y(y)$, where $r_X(x)$ and $r_Y(y)$ are the marginal failure rates.

When the joint survival function $S(x, y)$ has continuous second order partial derivatives at all points (x, y) in the first quadrant, the vector-valued function $\mathbf{R}(x, y) = (r_1(x, y), r_2(x, y))$ is called a *hazard gradient* of the random vector (X, Y). The components $r_1(x, y)$ and $r_2(x, y)$ can not be arbitrary and must be related by equation $\frac{\partial}{\partial y} r_1(x, y) = \frac{\partial}{\partial x} r_2(x, y)$ for all $x, y \geq 0$.

The hazard gradient vector $\mathbf{R}(x, y) = (r_1(x, y), r_2(x, y))$ uniquely determines (characterizes) the bivariate distribution by means of line integral through exponential representation

$$S(x, y) = \exp\left\{ - \int_{\mathcal{C}} \mathbf{R}(\mathbf{z}) \, d\mathbf{z} \right\}, \tag{1}$$

[1]kolev.ime@gmail.com

DOI: 10.5540/03.2023.010.01.0084

where $\mathcal{C}$ is any sufficiently smooth continuous path beginning at $(0,0)$ and terminating at (x,y). Note that "$d\mathbf{z}$" in (1) means that we are moving along the curve $\mathcal{C}$, instead on coordinate axes. The relation (1) holds provided that along the path of integration $S(x,y)$ is absolutely continuous, consult [5]. A discrete version of (1) was introduced in [3] where the discrete line integral is defined on an uniform grid with nodes (x,y) for $x,y=0,1,2\dots$

Line integrals of the type appearing in (1) can be evaluated by expressing a path $\mathcal{C}$ in a parametric form. Since the line integral of gradient vector does not depend on the path, one can arbitrarily choose particular interesting smooth continuous connecting paths $\mathcal{C}_1$ from $(0,0)$ to (x_0,y_0) and $\mathcal{C}_2$ from (x_0,y_0) to (x,y), such that $\int_{\mathcal{C}} \mathbf{R}(\mathbf{z})\,d\mathbf{z} = \int_{\mathcal{C}_1} \mathbf{R}(\mathbf{z})\,d\mathbf{z} + \int_{\mathcal{C}_2} \mathbf{R}(\mathbf{z})\,d\mathbf{z}$. This additive property can be applied to get an expression of the line integral in order to obtain via (1) the corresponding representation of the joint survival function $S(x,y)$ under the knowledge of analytical form of the components $r_1(x,y)$ and $r_2(x,y)$ of gradient vector $\mathbf{R}(x,y)$.

For instance, let $x \geq y \geq 0$ and consider a path $\mathcal{C}$ as a union of two line segments $\mathcal{C}_1$ and $\mathcal{C}_2$ linking the points $(0,0)$ and $(x-y,0)$ and the point $(x-y,0)$ with (x,y), correspondingly. These paths can be parametrized as follows

$$\mathcal{C}_1: \quad z_1(t)=t,\ z_2(t)=0 \quad \text{for} \quad t\in[0,x-y] \quad \text{with} \quad z_1'(t)=1, z_2'(t)=0$$

and

$$\mathcal{C}_2: \quad z_1(t)=x-y+t,\ z_2(t)=t \quad \text{for} \quad t\in[0,y] \quad \text{with} \quad z_1'(t)=z_2'(t)=1,$$

where $z_1'(t)$ and $z_2'(t)$ mean the corresponding first derivatives. Applying (1) leads to

$$S(x,y) = S_X(x-y)\exp\left\{-\int_0^y [r_1(x-y+t,t)+r_2(x-y+t,t)]dt\right\},$$

where $S_X(x)=P(X>x)$ is the marginal survival function of random variable X.

By analogy, we can compute $S(x,y)$ when $x \leq y$. We link both expressions as follows

$$S(x,y) = \begin{cases} S_X(x-y)\exp\left\{-\int_0^y [r_1(x-y+t,t)+r_2(x-y+t,t)]\,dt\right\} & \text{if } x\geq y\geq 0; \\ S_Y(y-x)\exp\left\{-\int_0^x [r_1(t,y-x+t)+r_2(t,y-x+t)]\,dt\right\} & \text{if } y\geq x\geq 0, \end{cases} \tag{2}$$

where $S_Y(y)=P(Y>y)$ is the survival function of random variable Y.

If $S(x,y)$ is differentiable at (x,y), the sum $r(x,y)=r_1(x,y)+r_2(x,y)$ is the directional derivative of $-\ln[S(x,y)]$ along the unit vector $(1,1)$ evaluated at $t=0$, i.e.,

$$r(x,y) = \lim_{t\to 0}\frac{-\ln S(x+t,y+t)}{t} = \frac{\partial}{\partial t}\left[-\ln S(x+t,y+t)\right]\Big|_{t=0}.$$

Therefore, the sum $r(x,y)$ is an important function to be used in a first step of modeling under the absence of information. This fact inspired the authors of [6] who characterized a family of bivariate continuous distributions such that the sum of components of the hazard vector has an additive decomposition, i.e., $r_1(x,y)+r_2(x,y)=a_0+a_1f(x)+a_2g(y)$, where $f(x)$ and $g(y)$ are non-negative continuous integrable functions with constants (parameters) $a_0,a_1,a_2\geq 0$. The corresponding class of bivariate distributions possessing such a property is huge. It has members that can be symmetric or asymmetric, absolutely continuous or with a singular component along the line $\{x=y\}$, positive or negative quadrant dependent.

Here, we will complement the study in [6] considering bivariate continuous distributions specified by relation

$$r(x,y)=r_1(x,y)+r_2(x,y)=A_0+A_1f(x)g(y) \quad \text{for} \quad x,y\geq 0, \tag{3}$$

with coefficients $A_0, A_1 \geq 0$, where $f(x)$ and $g(y)$ are non-negative continuous differentiable functions such that their product $f(x)g(y)$ is integrable. Let us denote by $\mathcal{L}(\mathbf{x})$ the class of bivariate continuous distributions generated by relation (3), with $\mathbf{x} = (x, y)$.

The exposition is organized as follows. In Section 2 we obtain the joint survival function of distributions belonging to the class $\mathcal{L}(\mathbf{x})$. We establish several equivalent representations and related functional equations. In Section 3 we consider a particular member of $\mathcal{L}(\mathbf{x})$ and fit the corresponding model generated by relation (3) to a Canadian joint life insurance data set analyzed by many authors. We finalize with conclusions.

2 Main properties of the class $\mathcal{L}(\mathbf{x})$

The class $\mathcal{L}(\mathbf{x})$ generated by relation (3) includes absolutely continuous distributions (if $A_0 = 0$, for example) and those having a singular component as well. Therefore, the corresponding joint survival function $S(x, y)$ is decomposable by an absolutely continuous component $S^{ac}(x, y)$ in the support of $\mathbf{R}^2_+ = \{(x, y)|x, y \geq 0\}$ and a singular one, $S^{si}(\max\{x, y\})$ with support on the set $\{(x, y) \in \mathbf{R}^2_+ | x = y\}$, i.e.,

$$S(x, y) = (1 - \alpha)S^{ac}(x, y) + \alpha S^{si}(\max\{x, y\}) \quad \text{for} \quad \alpha = P(X = Y) \in [0, 1]. \tag{4}$$

The bivariate survival function $S(x, y)$ in (4) is proper, i.e. when $\dfrac{\partial^2}{\partial x \partial y} S(x, y) \geq 0$, if and only if both $S^{ac}(x, y)$ and $S^{si}(\max\{x, y\})$ are valid survival functions.

We will derive first the expression for the joint survival function $S(x, y)$ of bivariate distributions belonging to the class $\mathcal{L}(\mathbf{x})$ and we will deduce when the joint distribution is absolutely continuous. It will be shown that $S(x, y)$ is a solution of a functional equation. As a result, we suggest algorithm to construct new distributions of the class $\mathcal{L}(\mathbf{x})$. Several particular members of $\mathcal{L}(\mathbf{x})$ are considered as well.

2.1 Joint survival function and closure properties

It follows our first characterization for distributions belonging to the class $\mathcal{L}(\mathbf{x})$ defined by (3).

Theorem 2.1. *The members of the class $\mathcal{L}(\mathbf{x})$ have a joint survival function $S(x, y)$ given by*

$$S(x, y) = \begin{cases} S_X(x - y) \exp\left[-A_0 y - A_1 \int_0^y f(x - y + t)g(t)dt\right], & \text{if } x \geq y \geq 0; \\ S_Y(y - x) \exp\left[-A_0 x - A_1 \int_0^x f(t)g(y - x + t)dt\right], & \text{if } y \geq x \geq 0, \end{cases} \tag{5}$$

if and only if relation (3) is satisfied.

Proof. Let (3) be fulfilled. Substitute $r(x, y) = A_0 + A_1 f(x)g(y)$ in (2) to get representation (5).

Conversely, let (5) be true. We will check that (3) is valid as well.

Take a ln in both sides of (5) and differentiate $-\ln[S(x, y)]$ with respect to x to obtain

$$r_1(x, y) = \begin{cases} r_X(x - y) + A_1 \int_0^y f'(x - y + t)g(t)dt, & \text{if } x \geq y; \\ -r_Y(y - x) + A_0 + A_1 f(x)g(y) - A_1 \int_0^x f(t)g'(y - x + t)dt, & \text{if } y \geq x, \end{cases}$$

where f' and g' are the first derivative of functions f and g.

Similarly, differentiation of $-\ln[S(x, y)]$ with respect to y leads to

$$r_2(x, y) = \begin{cases} -r_X(x - y) + A_0 + A_1 f(x)g(y) - A_1 \int_0^y f'(x - y + t)g(t)dt, & \text{if } x \geq y; \\ r_Y(y - x) + A_1 \int_0^x f(t)g'(y - x + t)dt, & \text{if } y \geq x. \end{cases}$$

Using these expressions of $r_1(x,y)$ and $r_2(x,y)$, it is direct to check that (3) is satisfied in all cases ($x > y$, $x < y$ and $x = y$). $\square$

Remark 2.1. (Bounds for marginal hazard rates): *The joint survival function $S(x,y)$ given by (5) is proper only for certain marginal distributions of X and Y.*

For $y \geq x$ in the proof of Theorem 2.1 we got that

$$r_1(x,y) = -r_Y(y-x) + A_0 + A_1 f(x)g(y) - A_1 \int_0^x f(t)g'(y-x+t)dt.$$

Since $r_1(x,y) \geq 0$ for all $x, y \geq 0$, the substitution $x = 0$ in last relation implies the upper bound for the marginal failure rate $r_Y(y) \leq A_0 + A_1 f(0)g(y)$.

By analogy, from inequality $r_2(x,y) \geq 0$ when $x \geq y$ we conclude that $r_X(x) \leq A_0 + A_1 f(x)g(0)$.

The restrictions on marginal failure rates in Remark 2.1 and condition $\frac{\partial^2}{\partial x \partial y} S(x,y) \geq 0$ determine the parameter space of models belonging to the class $\mathcal{L}(\mathbf{x})$.

Remark 2.2. (Closure properties of $\mathcal{L}(\mathbf{x})$): *Denote by $\mathcal{S}, \mathcal{S}_1$ and $\mathcal{S}_2$ survival functions of bivariate distributions belonging to the class $\mathcal{L}(\mathbf{x})$. It is readily verified that the following closure properties are fulfilled.*

(a) $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{L}(\mathbf{x})$, then their product $\mathcal{S}_1\mathcal{S}_2 \in \mathcal{L}(\mathbf{x})$;

(b) $\mathcal{S} \in \mathcal{L}(\mathbf{x})$, then $\mathcal{S}^c \in \mathcal{L}(\mathbf{x})$ for some $c \geq 1$;

(c) $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{L} \implies \mathcal{S}_1^{c_1}\mathcal{S}_2^{c_2} \in \mathcal{L}(\mathbf{x})$ for $c_1, c_2 \geq 1$;

(d) If $(X,Y) \in \mathcal{L}(\mathbf{x})$, then $S(cx, cy) \in \mathcal{L}(\mathbf{x})$ for $c > 0$.

These closure properties can be used to generate new members of the class $\mathcal{L}(\mathbf{x})$ (with more parameters) using as a base existing ones. Another building procedure is suggested in Remark 3.1.

We present without a proof a characterization theorem offering conditions that identify the absolutely continuous members of the class $\mathcal{L}(\mathbf{x})$, i.e., when $\alpha = P(X = Y) = 0$ in (4).

Theorem 2.2. *Consider the class $\mathcal{L}(\mathbf{x})$ specified by the joint survival function given in (5). Let the marginal distributions of X and Y have absolutely continuous density functions f_X and f_Y, respectively. The survival function $S(x,y)$ in (5) is absolutely continuous if and only if*

$$A_0 + A_1 f(0)g(0) = f_X(0) + f_Y(0).$$

2.2 Related functional equations

The next statement shows that the class $\mathcal{L}(\mathbf{x})$ can be characterized by a functional equation.

Theorem 2.3. *S(x,y) specified by (5) is a solution of the functional equation*

$$S(x+t, y+t) = S(x,y)S(t,t)\exp\left\{-A_1 \int_0^t [f(x+u)g(y+u) - f(u)g(u)]du\right\} \tag{6}$$

for all $x, y, t \geq 0$.

Proof. Let (5) be fulfilled. Assume that $x \geq y$ and apply (5) for $S(x+t, y+t), S(x,y)$ and $S(t,t)$ to confirm that relation (6) is true. The same conclusion is valid if $y \geq x$.

Conversely, let the functional equation (6) be true. Define a function $B(x,y)$, such that

$$S(x,y) = B(x,y)\exp\left\{-A_1 \int_0^x f(u)g(u)du - A_1 \int_0^y [f(u) - f(x-y+u)]g(u)du\right\}. \tag{7}$$

After some algebra one obtains the relation $B(x+t, y+t) = B(x,y)B(t,t)$ for all $x, y, t \geq 0$, which helps to arrive into to the first equation in (5) for $x \geq y$.

By analogy, one would get the second expression (5) when $y \geq x$. □

In the proof of Theorem 2.3 we concluded that, for all $x, y, t \geq 0$, the functional equation $B(x+t, y+t) = B(x,y)B(t,t)$ is fulfilled. Note that the function $B(x,y)$ is not necessarily a proper joint survival function. But, when it is, then the last functional equation generates bivariate distributions possessing the classical bivariate lack of memory property (BLMP), consult [4] to see a general representation of these distributions and examples. Therefore, one might use the following procedure to construct bivariate distributions belonging to the class $\mathcal{L}(\mathbf{x})$.

Remark 2.3. (Building procedure):

1. *Choose a bivariate distribution with joint survival function $S_1(x,y)$ satisfying the BLMP;*
2. *Based on relation (7), multiply $S_1(x,y)$ by the exponential expression to get a new member of the class $\mathcal{L}(\mathbf{x})$ with extended parameter space, containing parameters of $S_1(x,y)$.*

We will discuss one more classical functional equation related to the class $\mathcal{L}(\mathbf{x})$ and its particular cases. Let the functions f and g in (3) are positive and $A_1 > 0$.

Substitute $\psi(x,y) = r(x,y) - A_0 = A_1 f(x)g(y)$. Then, $f(x) = \dfrac{\psi(x,0)}{A_1 g(0)}$ and $g(y) = \dfrac{\psi(0,y)}{A_1 f(0)}$ leading to the functional equation

$$\psi(x,y)\psi(0,0) = \psi(x,0)\psi(0,y), \quad x, y \geq 0. \tag{8}$$

Case 1. We will search solutions of (8) specified by $\psi(x,y) = A_1 F(x+y)$, where F is a non-negative increasing continuous function. In such a case, (8) can be rewritten as a functional equation $F(x+y)F(0) = F(x)F(y)$ with non-trivial solutions $F(x) = e^{A(x)}$, where the function $A(x)$ is an arbitrary solution of additive equation $A(x+y) = A(x) + A(y)$, see Theorem 1.36 in [7], pages 27-28. Hence,

$$r(x,y) = A_0 + A_1 e^{ax+by} \quad \text{for some} \quad a > 0, b > 0. \tag{9}$$

Case 1A. Let $A_0 = 0$. In [2] is established that the bivariate Gompertz distribution

$$S(x,y) = \exp\{-c(e^{ax+by} - 1)\} \quad \text{for some} \quad a > 0, b > 0, c > 1 \tag{10}$$

is characterized by (9), with $A_0 = 0$ and $A_1 = c(a+b)$.

Thus, if $A_0 = 0$, $f(0) \neq 0$ and $g(0) \neq 0$, then the only possible functions in relation (3) are $f(x) = e^{ax}$ and $g(y) = e^{by}$ with $A_1 = c(a+b)$, leading to the bivariate Gompertz distribution.

Case 1B. Let $A_0 \neq 0$. Equation (9) is satisfied for a random vector (X,Y) generated by the stochastic representation

$$(X,Y) =^d (\min(Z,U), \min(T,V)), \tag{11}$$

where (Z,T) follows the bivariate Gompertz distribution specified by (10) and being independent of the vector (U,V) exhibiting BLMP with a singular component. In other words, the vector (X,Y) is bivariate Gompertz-Makeham distributed. We summarize the above findings in the next.

Corollary 2.1. *Consider a subclass $\mathcal{L}_1(\mathbf{x})$ of $\mathcal{L}(\mathbf{x})$ specified by $r(x,y) = A_0 + A_1 F(x+y)$ for $x, y \geq 0$, where F is a non-negative continuous increasing function. Then, the subclass $\mathcal{L}_1(\mathbf{x})$ has two main members: the bivariate Gompertz distribution given by (10), if $A_0 = 0$ and the bivariate Gompertz-Makeham distribution, if $A_0 > 0$.*

Example 2.1. *Suppose that (U,V) in (11) follows the classical bivariate exponential distribution with joint survival function* $\exp\{-\lambda_1 x-\lambda_2 y-\lambda_3\max(x,y)\}$ *for* $\lambda_1,\lambda_2,\lambda_3>0$. *Then, the joint survival function of the vector (X,Y) specified by (11) is given by*

$$S(x,y)=\exp\{-c(e^{ax+by}-1)-\lambda_1 x-\lambda_2 y-\lambda_3\max(x,y)\}. \tag{12}$$

Case 2. Let $f(0)=g(0)=0$ and $A_1>0$ in (3). Assume that $r(x,y)-A_0=G(xy)$, where G is a non-negative increasing continuous function. Then the functional equation $G(xy)=A_1f(x)g(y)$ is a Pexider equation with non-trivial solution $G(x)=A_1x^c$ for $c>0$, see Result 1.56(d) in [7]. Thus, $f(x)=ax^c$ and $g(y)=by^c$, with $A_1=ab$ and $c>0$. Therefore, we have

Corollary 2.2. *Consider a subclass $\mathcal{L}_2(\mathbf{x})$ of $\mathcal{L}(\mathbf{x})$ specified by $r(x,y)=A_0+A_1G(xy)$ for $x,y\geq 0$, where G is a non-negative continuous increasing function. Then, the distributions of the subclass $\mathcal{L}_2(\mathbf{x})$ are such that $r(x,y)=A_0+A_1(xy)^c$ for some positive constant c.*

Example 2.2. *Let c= 1 in Corollary 2.2, i.e., $r(x,y)=r_1(x,y)+r_2(x,y)=A_0+A_1xy$. Then $f(x)=x, g(y)=y$ and from (5) we obtain*

$$S(x,y)=\begin{cases} S_X(x-y)\exp\left\{-A_0y-\frac{A_1(x-y)y^2}{2}-\frac{A_1y^3}{3}\right\}, & \text{if } x\geq y;\\ S_Y(y-x)\exp\left\{-A_0x-\frac{A_1(y-x)x^2}{2}-\frac{A_1x^3}{3}\right\}, & \text{if } x\leq y.\end{cases} \tag{13}$$

The functional equation (6) in our case transforms into

$$S(x+t,y+t)=S(x,y)S(t,t)\exp\left\{-A_1\left[xyt+\frac{(x+y)t^2}{2}\right]\right\}. \tag{14}$$

Therefore, due to Theorem 2.3 relations (13) and (14) are equivalent.

The building procedure from Remark 2.3 reads as follows:

1) Choose bivariate distribution possesing the BLMP;

2) Multiply the corresponding joint survival function by $\exp\left[-\frac{A_1x^3}{3}+\frac{A_1(x-y)y^2}{2}\right]$ *to get new member of the class $\mathcal{L}_2(\mathbf{x})$ when $c=1$.*

3 Joint life insurance application

As an application, we examine a sample of censored residual lifetimes of couples of insureds extracted from a data set of annuities contracts of a Canadian life insurance company, registered in the period from December 29, 1988 to December 31, 1993. The data set is both left and right truncated. The available information provides the entry ages of the two spouses and the corresponding censored residual lifetimes. The Canadian data set has already been analyzed by many authors, consult [1].

We considered contracts with entry ages greater than 60 only, for a total number of observations equal to 9535. It is convenient to fit the data with the model (12) belonging to the subclass $\mathcal{L}_1(\mathbf{x})$ since it includes a possibility of common external shocks, governed by parameter λ_3. Thus, we postulate marginal distributions of Gompertz-Makeham type (mainly used in actuarial practice).

We used the two-stage maximum likelihood technique: first we compute the maximum likelihood estimates of the parameters of the marginal distributions (i.e., $\hat{a},\hat{\lambda_1}$ and $\hat{b},\hat{\lambda_2}$), and then we obtain the maximum likelihood estimates of the remaining dependence parameters $\hat{c}$ and $\hat{\lambda_3}$ assuming those already estimated as given. We did goodness of fit comparison with other models in literature through the Bayesian Information Criteria (BIC) concluding that the model (12) outperforms the existing ones. For example, the best fit reported in [1] based on the Extended Marshall-Olkin model produces $BIC = 3088,09$ while the use of model (12) implies $BIC = 2732,12$.

Many joint life actuarial products depend on the residual lifetimes joint survival distribution values on the straight line $\{x = y\}$. For instance, the continuous n-years joint life annuity net premium is defined by $\int_0^n exp(-ku)S(u,u)du$, where k is the instantaneous interest rate. It has been calculated in our case, offering an attractive alternative for actuarial practice, where for simplicity the independence assumption is usually adopted.

4 Conclusions

In this note we show how to create bivariate continuous probability models through the exponential representation (1) of the joint survival function involving line integral by taking into account the physical nature of the data analyzed. We study the class $\mathcal{L}(\mathbf{x})$ generated by the relation (3) containing many classical and new bivariate models. We established several characterization theorems and equivalent relations (consult Theorem 2.1, Theorem 2.3 and functional equation (8)) and offer building procedures. The method proposed is a new powerful tool and might be successfully extended to the multivariate case for the needs of data analysis of big data sets. An interesting problem for a future research is to define, study and apply a discrete version of the class $\mathcal{L}(\mathbf{x})$ on uniform and logically connected grids.

Acknowledgements

The author is partially supported by FAPESP grants 2013/07375-0 and 2021/14790-0.

I wish to thank the Society of Actuaries, through the courtesy of Edward (Jed) Frees and Emiliano Valdez, for allowing the use of the data in this paper.

References

[1] Fabio Gobbi, Nikolai Kolev, and Sabrina Mulinacci. "Joint life insurance pricing using extended Marshall–Olkin models". In: **ASTIN Bulletin: The Journal of the IAA** 49.2 (2019), pp. 409–432. DOI: 10.1017/asb.2019.3.

[2] Nikolai Kolev. "Characterizations of the class of bivariate Gompertz distributions". In: **Journal of Multivariate Analysis** 148 (2016), pp. 173–179. DOI: 10.1016/j.jmva.2016.03.004.

[3] Nikolai Kolev. "Discrete line integral on uniform grids: Probabilistic interpretation and applications". In: **Brazilian Journal of Probability and Statistics** 34.4 (2020), pp. 821–843. DOI: 10.1214/19-BJPS454.

[4] H. V. Kulkarni. "Characterizations and Modelling of Multivariate Lack of Memory Property". In: **Metrika** 64.2 (2006), pp. 167–180. DOI: 10.1007/s00184-006-0042-2.

[5] Albert W. Marshall. "Some comments on the hazard gradient". In: **Stochastic Processes and their Applications** 3.3 (1975), pp. 293–300. DOI: 10.1016/0304-4149(75)90028-9.

[6] Jayme Pinto and Nikolai Kolev. "Sibuya-type bivariate lack of memory property". In: **Journal of Multivariate Analysis** 134 (2015), pp. 119–128. DOI: 10.1016/j.jmva.2014.11.001.

[7] Prasanna K. Sahoo and Palaniappan Kannappan. **Introduction to Functional Equations**. Chapman and Hall/CRC, Feb. 2011. DOI: 10.1201/b10722.

