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Second-Order Geometric Characterization of Optimal Solutions in Continuous-Time Programming

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Resumo. In this work, it is properly defined second-order tangent directions, second-order feasible directions and second-order directions of decrease for continuous-time nonlinear programming. In addition, it is established necessary optimality conditions in geometric form.

Palavras-chave. Continuous-Time Programming, Necessary Optimality Conditions, Geometric Characterization.

1 Introduction

The paper addresses the following continuous-time nonlinear programming problem:

$$\begin{array}{ll} \text{minimize} & F(z) = \displaystyle\int_0^T f(z(t),t)dt \\ \text{subject to} & g(z(t),t) \le 0 \text{ a.e. in } [0,T], \\ & z \in L^\infty([0,T];\mathbb{R}^n), \end{array} \tag{CTP}$$

where $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ and $g : \mathbb{R}^n \times [0,T] \to \mathbb{R}^m$. Second-order necessary optimality conditions for this kind of problems were derived just recently in the literature in Monte and de Oliveira [2, 3]. In these papers, a more general problem, with equality and inequality constraints, is considered. The second-order necessary optimality conditions were established by means of an indirect analytic approach. The results are obtained, at first, for the problems with equality constraints only, where one of the main tools is the Uniform Implicit Function Theorem (see Pinho [4]). Then, by adding slackness variables, the inequality constraints are transformed in equality ones and the previous result is applied. This indirect approach has the characteristic that the regularity condition involves not only the jacobian of the active constraints but the matrix $\left[\nabla g(\bar{z}(t),t) \quad \text{diag}\{-2\bar{w}_j(t)\}_{j=1}^m\right]$, where $\bar{w}_j(t) = \sqrt{-g_j(\bar{z}(t),t)}$. It would be interesting to derive optimality conditions in which the regularity condition involves only the vectors $\nabla g_j(\bar{z}(t),t)$, with j related to the active constraint.

Here, we follow a different path. The necessary optimality conditions presented in this work are deduced using a geometric approach. We use the concepts of tangent directions, feasible directions, and descent directions. We show that the functional F does not admit a descent direction that is, at the same time, also tangent to the feasible set Ω. We believe that this geometric approach is more adequate for obtaining necessary optimality conditions under weaker and natural regularity conditions.

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2 Preliminaries

The set of all feasible solutions will be denoted by

$$\Omega = \{z \in L^\infty([0,T];\mathbb{R}^n) : g(z(t),t) \leq 0 \text{ a.e. in } [0,T]\}.$$

We denote $I := \{1,\dots,m\}$. The index set of the active constraints at instant t at $\bar{z} \in \Omega$ will be denoted by $I_a(t)$, that is,

$$I_a(t) = \{i \in I : g_i(\bar{z}(t),t) = 0\}.$$

$\bar{B}$ denotes the closed unit ball centered at the origin in $\mathbb{R}^n$.

A feasible solution $\bar{z} \in \Omega$ is said to be a *local optimal solution* of (CTP) if there exists $\varepsilon > 0$ such that $F(\bar{z}) \leq F(z)$ for all $z \in \Omega$ satisfying $z(t) \in \bar{z}(t) + \varepsilon\bar{B}$ a.e. in $[0,T]$.

We say that the basic assumptions are valid at $\bar{z} \in \Omega$ if there exists $\varepsilon > 0$ such that

(H1) $f(z,\cdot)$ is measurable for each z;

$f(\cdot,t)$ is twice continuously differentiable in $\bar{z}(t) + \varepsilon\bar{B}$ a.e. in $[0,T]$;

there exits $K_f > 0$ such that

$$|\nabla f(\bar{z}(t),t)| + |\nabla^2 f(\bar{z}(t),t)| \leq K_f \text{ a.e. in } [0,T];$$

(H2) $g(z,\cdot)$ is measurable for each z;

$g(\cdot,t)$ is twice continuously differentiable in $\bar{z}(t) + \varepsilon\bar{B}$, uniformly in t, a.e. in $[0,T]$;

$g(z(\cdot),\cdot)$ is essentially bounded in $[0,T]$ for each $z \in \Omega$ such that $\|z - \bar{z}\|_\infty < \varepsilon$;

there exists $K_g > 0$ such that

$$|\nabla g(\bar{z}(t),t)| + |\nabla^2 g(\bar{z}(t),t)| \leq K_g \text{ a.e. in } [0,T].$$

3 Main Results

This section is devoted to the main results. Next, we have the definition of second-order tangent directions.

Definition 3.1. *Let $\bar{z},\gamma,\zeta \in L^\infty([0,T];\mathbb{R}^n)$. It is said that ζ is a second-order tangent direction of Ω at $\bar{z}$ with respect to γ if there exist sequences $\{z^k\}_{l=1}^\infty \subset \Omega$ and $\{\alpha_k\}_{l=1}^\infty \subset \mathbb{R}$ with $\alpha_k > 0$ for all k such that $z^k \to \bar{z}$ and $\alpha_k \to 0$, when $k \to \infty$, and*

$$\lim_{k\to\infty} \frac{z^k - \bar{z} - \alpha_k\gamma}{\alpha_k^2} = \zeta.$$

The set of all second-order tangent directions of Ω at $\bar{z}$ with respect to γ will be denoted by $\mathscr{T}^2_\Omega(\bar{z},\gamma)$.

The set of second order tangent directions is not always a cone. Also, it may not be convex. But it has the property stated in the proposition below.

Proposition 3.1. *Let $\bar{z},\gamma \in L^\infty([0,T];\mathbb{R}^n)$. The set $\mathscr{T}^2_\Omega(\bar{z},\gamma)$ is closed.*

Proof. Let $\{\zeta^l\}_{l=1}^{\infty} \subset \mathscr{T}_\Omega^2(\bar{z},\gamma)$ be a sequence such that $\zeta^l \to \zeta$ when $l \to \infty$. We shall show that $\zeta \in \mathscr{T}_\Omega^2(\bar{z},\gamma)$. By the fact that $\{\zeta^l\} \subset \mathscr{T}_\Omega^2(\bar{z},\gamma)$ for all l, it follows from the definition that, for each l, there exist sequences $\{z^{l,k}\}_{k=1}^{\infty} \subset \Omega$ and $\{\alpha_{l,k}\}_{k=1}^{\infty} \subset \mathbb{R}$ with $\alpha_{l,k} > 0$ for all k such that $z^{l,k} \to \bar{z}$ and $\alpha_{l,k} \to 0$, when $k \to \infty$, and

$$\lim_{k\to\infty} \frac{z^{l,k} - \bar{z} - \alpha_{l,k}\gamma}{\alpha_{l,k}^2} = \zeta^l.$$

Let $\{\varepsilon_l\}_{l=1}^{\infty}$ be any sequence with $\varepsilon_l \to 0$ when $l \to \infty$. For each l, we can find $k(l)$ big enough such that

$$\left\| \frac{z^{l,k(l)} - \bar{z} - \alpha_{l,k(l)}\gamma}{\alpha_{l,k(l)}^2} - \zeta^l \right\|_{\infty} \le \varepsilon_l.$$

Thence,

$$\left\| \frac{z^{l,k(l)} - \bar{z} - \alpha_{l,k(l)}\gamma}{\alpha_{l,k(l)}^2} - \zeta \right\|_{\infty} \le \varepsilon_l + \|\zeta^l - \zeta\|_{\infty}.$$

Therefore, $\{z^{l,k(l)}\}_{l=1}^{\infty} \subset \Omega$, $\{\alpha_{l,k(l)}\}_{l=1}^{\infty} \subset \mathbb{R}$ with $\alpha_{l,k(l)} > 0$ for all k, $z^{l,k(l)} \to \bar{z}$ and $\alpha_{l,k(l)} \to 0$, when $k \to \infty$, and

$$\lim_{k\to\infty} \frac{z^{l,k(l)} - \bar{z} - \alpha_{l,k(l)}\gamma}{\alpha_{l,k(l)}^2} = \zeta.$$

Thus, $\zeta \in \mathscr{T}_\Omega^2(\bar{z},\gamma)$. Then, $\mathscr{T}_\Omega^2(\bar{z},\gamma)$ is closed. $\square$

Below, we define a set that can be seen as a set of second-order feasible directions.

Definition 3.2. *Let $\bar{z},\gamma \in L^\infty([0,T];\mathbb{R}^n)$. One define $\mathscr{F}_\Omega^2(\bar{z},\gamma)$ as the set of all direction $\zeta \in L^\infty([0,T];\mathbb{R}^n)$ such that*

$$\nabla g_i(\bar{z}(t),t)^\top \zeta(t) + \frac{1}{2}\gamma(t)^\top \nabla^2 g_i(\bar{z}(t),t)\gamma(t) \le 0,\ i \in I_a(t),\ \text{a.e. in } [0,T].$$

Proposition 3.2. *Let $\bar{z} \in \Omega$. Assume that (H2) holds for some $\varepsilon > 0$. If $\gamma \in L^\infty([0,T];\mathbb{R}^n)$ is such that*

$$\nabla g_i(\bar{z}(t),t)^\top \gamma(t) = 0,\ i \in I_a(t),\ \text{a.e. in } [0,T],$$

then

$$\mathscr{T}_\Omega^2(\bar{z},\gamma) \subset \mathscr{F}_\Omega^2(\bar{z},\gamma).$$

Proof. Let $\zeta \in \mathscr{T}_\Omega^2(\bar{z},\gamma)$. Then, there exist sequences $\{z^k\}_{l=1}^{\infty} \subset \Omega$ and $\{\alpha_k\}_{l=1}^{\infty} \subset \mathbb{R}$ with $\alpha_k > 0$ for all k such that $z^k \to \bar{z}$ and $\alpha_k \to 0$, when $k \to \infty$, and

$$\lim_{k\to\infty} \frac{z^k - \bar{z} - \alpha_k\gamma}{\alpha_k^2} = \zeta.$$

We will denote

$$\zeta^k = \frac{z^k - \bar{z} - \alpha_k\gamma}{\alpha_k^2},$$

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so that $z^k = \bar{z} + \alpha_k \gamma + \alpha_k^2 \zeta^k$ for all k and $\zeta^k \to \zeta$ when $k \to \infty$. By Taylor's expansion, for each k and for $i \in I_a(t)$,

$$\begin{aligned} g_i(z^k(t),t) &= g_i(\bar{z}(t),t) + \nabla g_i(\bar{z}(t),t)^\top [z^k(t) - \bar{z}(t)] \\ &\quad + \frac{1}{2}[z^k(t) - \bar{z}(t)]^\top \nabla^2 g_i(\bar{z}(t),t)[z^k(t) - \bar{z}(t)] + o(\|z^k(t) - \bar{z}(t)\|^2) \\ &= \nabla g_i(\bar{z}(t),t)^\top \Big[\alpha_k \gamma(t) + \alpha_k^2 \zeta^k(t)\Big] \\ &\quad + \frac{1}{2}\Big[\alpha_k \gamma(t) + \alpha_k^2 \zeta^k(t)\Big]^\top \nabla^2 g_i(\bar{z}(t),t)\Big[\alpha_k \gamma(t) + \alpha_k^2 \zeta^k(t)\Big] + o(\alpha_k^2) \\ &= \alpha_k^2 \Big[\nabla g_i(\bar{z}(t),t)^\top \zeta^k(t) + \frac{1}{2}\gamma(t)^\top \nabla^2 g_i(\bar{z}(t),t)\gamma(t)\Big] + o(\alpha_k^2), \end{aligned}$$

for almost all $t \in [0,T]$, where $o(\alpha_k^2)/\alpha_k^2 \to 0$ when $k \to \infty$ and we have used the facts that $g_i(\bar{z}(t),t) = 0$ for $i \in I_a(t)$ a.e. in $[0,T]$ and that, by hypothesis, $\nabla g_i(\bar{z}(t),t)^\top \gamma(t) = 0$, $i \in I_a(t)$ a.e. in $[0,T]$. (Let us notice that, in fact, $o(\|z^k(t) - \bar{z}(t)\|^2)) = o(\alpha_k^2)$ for almost all $t \in [0,T]$, for

$$\begin{aligned} \frac{o(\|z^k(t) - \bar{z}(t)\|^2)}{\alpha_k^2} &= \frac{o(\|z^k(t) - \bar{z}(t)\|^2)}{\|z^k(t) - \bar{z}(t)\|^2} \cdot \frac{\|z^k(t) - \bar{z}(t)\|^2}{\alpha_k^2} \\ &= \frac{o(\|z^k(t) - \bar{z}(t)\|^2)}{\|z^k(t) - \bar{z}(t)\|^2} \cdot \frac{\|\alpha_k \gamma(t) + \alpha_k^2 \zeta^k(t)\|^2}{\alpha_k^2} \\ &= \frac{o(\|z^k(t) - \bar{z}(t)\|^2)}{\|z^k(t) - \bar{z}(t)\|^2} \cdot \frac{\alpha_k^2 \|\gamma(t) + \alpha_k \zeta^k(t)\|^2}{\alpha_k^2}, \end{aligned}$$

and $z^k(t) \to \bar{z}(t)$ and $\zeta^k(t) \to \zeta(t)$ uniformly for almost all $t \in [0,T]$, as $k \to \infty$.) Provided $z^k \in \Omega$ for all k, we have, for each $i \in I_a(t)$,

$$\alpha_k^2 \Big[\nabla g_i(\bar{z}(t),t)^\top \zeta^k(t) + \frac{1}{2}\gamma(t)^\top \nabla^2 g_i(\bar{z}(t),t)\gamma(t)\Big] + o(\alpha_k^2) \leq 0 \text{ a.e. in } [0,T]$$

for all k. Dividing the expression above by α_k^2 and taking limit with $k \to \infty$, the result follows. $\square$

The set defined below can be seen as the set of second-order descent directions of the functional F.

Definition 3.3. *Let $\bar{z}, \gamma \in L^\infty([0,T];\mathbb{R}^n)$. One define $\mathscr{A}_F^2(\bar{z},\gamma)$ as the set of all directions $\zeta \in L^\infty([0,T];\mathbb{R}^n)$ which satisfy*

$$\int_0^T [\nabla f(\bar{z}(t),t)^\top \zeta(t) + \frac{1}{2}\gamma(t)^\top \nabla^2 f(\bar{z}(t),t)\gamma(t)]dt < 0.$$

Proposition 3.3. *Let $\bar{z} \in \Omega$ be a local optimal solution of (CTP). Assume that (H1) and (H2) hold for some $\varepsilon > 0$. If $\gamma \in L^\infty([0,T];\mathbb{R}^n)$ is such that*

$$\int_0^T \nabla f(\bar{z}(t),t)^\top \gamma(t) dt \leq 0,$$

then

$$\mathscr{A}_F^2(\bar{z},\gamma) \cap \mathscr{T}_\Omega^2(\bar{z},\gamma) = \emptyset.$$

Proof. Assume that there exists $\zeta \in \mathscr{A}_F^2(\bar{z},\gamma) \cap \mathscr{T}_\Omega^2(\bar{z},\gamma)$. Then,

$$\int_0^T [\nabla f(\bar{z}(t),t)^\top \zeta(t) + \frac{1}{2}\gamma(t)^\top \nabla^2 f(\bar{z}(t),t)\gamma(t)]dt < 0$$

and there exist sequences $\{z^k\}_{l=1}^{\infty} \subset \Omega$ and $\{\alpha_k\}_{l=1}^{\infty} \subset \mathbb{R}$ with $\alpha_k > 0$ for all k such that $z^k \to \bar{z}$ and $\alpha_k \to 0$, when $k \to \infty$, and

$$\lim_{k\to\infty} \frac{z^k - \bar{z} - \alpha_k\gamma}{\alpha_k^2} = \zeta.$$

As in the previous proof, let us denote

$$\zeta^k = \frac{z^k - \bar{z} - \alpha_k\gamma}{\alpha_k^2},\ k = 1, \ldots, \infty.$$

It is clear that $\zeta^k \to \zeta$ in $L^\infty([0,T];\mathbb{R}^n)$, so that $\{\zeta^k\}_{k=1}^{\infty}$ is a bounded sequence (once it is convergent) in $L^\infty([0,T];\mathbb{R}^n)$. Let us say that $\|\zeta^k\|_\infty \le K_\zeta$ for all k, where $K_\zeta > 0$. By Taylor's expansion, we have, for each k, that

$$\begin{aligned}
F(z^k) &= F(\bar{z}) + \delta F(\bar{z}; z^k - \bar{z}) + \frac{1}{2}\delta^2 F(\bar{z}; (z^k - \bar{z}, z^k - \bar{z})) + o(\|z^k - \bar{z}\|_\infty^2) \\
&= F(\bar{z}) + \delta F(\bar{z}; \alpha_k\gamma + \alpha_k^2\zeta^k) \\
&\quad + \frac{1}{2}\delta^2 F(\bar{z}; (\alpha_k\gamma + \alpha_k^2\zeta^k, \alpha_k\gamma + \alpha_k^2\zeta^k)) + o(\|z^k - \bar{z}\|_\infty^2) \\
&= F(\bar{z}) + \alpha_k \int_0^T \nabla f(\bar{z}(t),t)^\top \gamma(t)dt \\
&\quad + \alpha_k^2 \int_0^T \Big[\nabla f(\bar{z}(t),t)^\top \zeta^k(t) + \frac{1}{2}\gamma(t)^\top \nabla^2 h_j(\bar{z}(t),t)\gamma(t)\Big] dt + o(\alpha_k^2) \\
&\le F(\bar{z}) + \alpha_k^2 \int_0^T \Big[\nabla f(\bar{z}(t),t)^\top \zeta^k(t) + \frac{1}{2}\gamma(t)^\top \nabla^2 h_j(\bar{z}(t),t)\gamma(t)\Big] dt + o(\alpha_k^2),
\end{aligned}$$

where we have used the hypothesis that $\int_0^T \nabla f(\bar{z}(t),t)^\top \gamma(t)dt \le 0$. Thence, by noticing that $|\nabla f(\bar{z}(t),t)^\top \zeta^k(t)| \le K_f K_\zeta$ a.e. in $[0,T]$ and using the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{k\to\infty} \frac{F(z^k) - F(\bar{z})}{\alpha_k^2} \le \int_0^T \Big[\nabla f(\bar{z}(t),t)^\top \zeta(t) + \frac{1}{2}\gamma(t)^\top \nabla^2 h_j(\bar{z}(t),t)\gamma(t)\Big] dt < 0,$$

so that $F(z^k) < F(\bar{z})$ for k big enough. This contradicts the optimality of $\bar{z}$. $\square$

It follows directly from the last two propositions the following result.

Corollary 3.1. *Let $\bar{z} \in \Omega$ be a local optimal solution of (CTP) and $\gamma \in L^\infty([0,T];\mathbb{R}^n)$ such that*

$$\int_0^T \nabla f(\bar{z}(t),t)^\top \gamma(t)dt \le 0$$

and

$$\nabla g_i(\bar{z}(t),t)^\top \gamma(t) = 0,\ i \in I_a(t),\ a.e.\ in\ [0,T].$$

Assume that (H1) and (H2) hold for some $\varepsilon > 0$. If $\mathscr{T}_\Omega^2(\bar{z},\gamma) \supset \mathscr{F}_\Omega^2(\bar{z},\gamma)$, then

$$\mathscr{A}_F^2(\bar{z},\gamma) \cap \mathscr{F}_\Omega^2(\bar{z},\gamma) = \emptyset.$$

4 Final Considerations

By making use of the previous geometric characterization of optimal solutions, by means of an adequate alternative theorem and by defining a regularity condition appropriately, we may obtain Karush-Kuhn-Tucker-type optimality conditions as stated next.

Theorem 4.1. *Let $\bar{z} \in \Omega$ be a local optimal solution of (CTP). Assume that some adequate constraint qualification is valid at $\bar{z}$. Assume also that (H1) and (H2) hold for some $\varepsilon > 0$. Let $\gamma \in L^\infty([0,T];\mathbb{R}^n)$ be such that*

$$\int_0^T \nabla f(\bar{z}(t),t)^\top \gamma(t)dt \leq 0 \text{ and } \nabla g_i(\bar{z}(t),t)^\top \gamma(t) = 0,\ i \in I_a(t),\ \text{a.e. in } [0,T].$$

Then there exists a multiplier $u \in L^\infty([0,T];\mathbb{R}^p)$ such that

$$\nabla f(\bar{z}(t),t) + \sum_{j \in I} u_i(t) \nabla g_i(\bar{z}(t),t) = 0 \text{ a.e. in } [0,T],$$
$$u_i(t) \geq 0,\ u_i(t) g_i(\bar{z}(t),t) = 0 \text{ a.e. in } [0,T],\ i \in I,$$

and

$$\int_0^T \gamma(t)^\top \left[\nabla^2 f(\bar{z}(t),t) + \sum_{i \in I} u_i(t) \nabla^2 g_i(\bar{z}(t),t)\right]\gamma(t)dt \geq 0.$$

The alternative theorem given in Arutyunov [1] may be an option. However, such a theorem requires a regularity condition which involves the objective along with the inequality constraints. This is not a genuine constraint qualification, once it involves the objective function. So, as a topic of future work, we will investigate a proper constraint qualification and an alternative theorem.

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