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Influence of asynchronous parametric excitation in stability maps of the simplest electromechanical system

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Abstract. In this paper the influence of asynchronous parametric excitation in stability maps of the simplest electromechanical system is analyzed. The system is composed by two interacting subsystems, a mechanical and an electromagnetic, and it has the minimum number of elements necessary to be classified as an electromechanical system. The system does not have elements that can store potential energies, neither mechanical nor electromagnetic. The system dynamics is written in terms of 2×2 inertia matrix M and gyroscopic matrix G. Two parametric excitation terms are introduced in G. The terms have an amplitude ϵ, frequency Ω and asynchrony with respect to each other θ. For different values of θ, stability maps, in terms of ϵ and Ω, are constructed for the electromechanical system with the parametric excitation. In each map, it can be seen stability and instability regions of the trivial solution (system's equilibrium) of the system. The objective of the paper is to analyze how the value of θ affects these stability and instability regions.

Palavras-chave. Electromechanical systems, asynchronous parametric excitation, stability maps.

1 Introduction

Parametric excitation in dynamical systems is caused by the presence of periodically varying system parameters, for example, stiffness or inertia, expressed through time-periodic coefficients in the equations of motion. The presence of these terms can cause destabilization or also stabilization of the trivial solution (system's equilibrium).

Several practical applications involving parametric excitation appeared over the last years. The destabilizing effects can be used in energy harvesting applications [7] and in parametric amplifiers. On the other hand, the anti-resonance effect is introduced in order to attenuate vibrations and to enhance dissipative properties. Another growing field of application is the microelectromechanical systems (MEMS). In this field, applications employing both stabilizing and destabilizing effects can be found.

Despite all the theoretical studies and practical applications, there still remain gaps concerning the studies on more general systems, especially those where the parametric excitation is not synchronous, i.e., when the individual system parameters have variation though of the same frequency, but with a phase shift [2, 3].

For example, it is known that asynchronous excitation can lead to a phenomena called total instability, which makes the trivial solution unstable for all excitation frequencies. This contrasts to other known parametric resonance cases which are limited to rather narrow frequency ranges. This global effect of total instability has not been studied in detail for the general case so far. Therefore there is a special interest in exploring the stability of asynchronous parametric excitation. We

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believe that such studies might improve the performance of existing applications or even foster new ones.

In this paper, the influence of asynchronous parametric excitation in stability maps of the simplest electromechanical system is analyzed. The system is composed by two interacting subsystems, a mechanical and an electromagnetic one, and it has the minimum number of elements necessary to be classified as an electromechanical system. The dynamic behavior of the electromechanical system depends on this mutual interaction, i.e., the phenomena present in the system response reflect this interplay between the mechanical and electromagnetic subsystems [5]. The system is composed by a DC motor connected to a rigid disc, a motor-disc system, and does not have elements that can store potential energies, neither mechanical nor electromagnetic.

The dynamics of the electromechanical system is parametrized with a mechanical (position of the disc) and an electromagnetic variable (charge in the DC motor). Since these variables are native and intrinsic to the problem, they are the most natural variables to parametrize the system dynamics. With such kind of variables, the set of differential equations present in the initial value problem that characterizes the system dynamics is a coupled set of equations [4].

The system dynamics is written in terms of 2×2 inertia matrix M and gyroscopic matrix G. Two parametric excitation terms are introduced in G. The terms have an amplitude ϵ, frequency Ω and asynchrony with respect to each other θ. For different values of θ, stability maps, in terms of ϵ and Ω, are constructed for the electromechanical system with the parametric excitation. In each map, it can be seen stability and instability regions of the trivial solution (system's equilibrium) of the system. The objective of the paper is to analyze how the value of θ affects these stability and instability regions.

2 Dynamics of the electromechanical system

The electromechanical system analyzed in this paper is a DC motor connected to a disc as shown in Fig. 1.

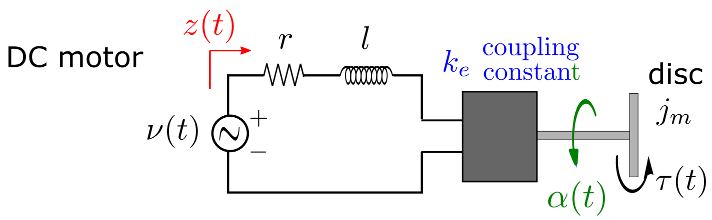

Figure 1: Electromechanical system.

The initial value problem that characterizes the system dynamics is defined as follows. Find $(\alpha,\ z)$ such that, for all $t > 0$,

$$
\begin{aligned}
l\ddot{z}(t) + r\,\dot{z}(t) + k_e\dot{\alpha}(t) &= \nu(t)\,,\\
j_m\ddot{\alpha}(t) + b_m\dot{\alpha}(t) - k_e\dot{z}(t) &= \tau(t)\,,
\end{aligned}
\tag{1}
$$

with the initial conditions $\dot{\alpha}(0) = \beta_0$, $\alpha(0) = \alpha_0$, $\dot{z}(0) = c_0$ and $z(0) = z_0$. In these equations, t is the time, ν is the source voltage, z is the electric charge, $\dot{\alpha}$ is the angular speed of the disc, l is the electric inductance, j_m is the disc moment of inertia, b_m is the damping ratio in the transmission of the torque generated by the motor, k_e is the motor electromagnetic force constant, r is the electrical resistance, and τ is an external torque made over the disc.

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The system state is given by four variables, two of them mechanical (angular velocity and position of the disc) and two of them electromagnetic (charge and current in the motor). These four variables are native and intrinsic to the problem, natural variables to parametrize the system state. The system dynamics, parametrized with these four variables, is given by an initial value problem comprising a set of two coupled differential equations. The coupling between the mechanical and electromagnetic subsystems is not given by a functional relation. It depends on the system state and, consequently, depends on initial conditions. Writing Eq. (1) in matrix form, and assuming $b_m = 0$, $r = 0$, $\nu = 0$ and $\tau = 0$ to get a conservative system, i.e., the simplest electromechanical system, we obtain:

$$\begin{bmatrix} l & 0 \\ 0 & j_m \end{bmatrix} \begin{bmatrix} \ddot{z}(t) \\ \ddot{\alpha}(t) \end{bmatrix} + \begin{bmatrix} 0 & k_e \\ -k_e & 0 \end{bmatrix} \begin{bmatrix} \dot{z}(t) \\ \dot{\alpha}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2}$$

$$M\ddot{\mathbf{Y}}(t) + G\dot{\mathbf{Y}}(t) = \mathbf{0}, \tag{3}$$

where M and G will be called inertia and gyroscopic matrices respectively and $\mathbf{Y} = \begin{bmatrix} z \\ \alpha \end{bmatrix}$. The initial conditions become $\dot{\mathbf{Y}}(0) = \begin{bmatrix} c_0 \\ \beta_0 \end{bmatrix}$ and $\mathbf{Y}(0) = \begin{bmatrix} z_0 \\ \alpha_0 \end{bmatrix}$. Matrix G is skew symmetric, i.e., $G^\star = -G$, where $\square^\star$ indicates Hermitian matrix. It is interesting to notice that despite calling M and G inertia and gyroscopic matrices, an usual terminology used in mechanical systems [6], here these two matrices have a different physical interpretation from the traditional one. M is not an inertia matrix like those that appear in purely mechanical systems. M is composed by elements that represent inertia of two different natures, a mechanical and an electromagnetic. G is also not a traditional gyroscopic matrix. In purely mechanical systems, a gyroscopic matrix usually couples motions in different directions. Here G couples the mechanical and electromagnetic subsystems. It is responsible for the interplay of energies between these two subsystems.

Besides having inertia and gyroscopic matrices with different physical interpretation, Eq. (3) has another big difference from what is found in dynamics of purely mechanical systems. It does not have a matrix composed by elements that can store potential energies to be called stiffness matrix. Our system system has neither elements that can store mechanical potential energy, as springs, nor elements that can store electric energy, as capacitors. Its dynamics equation is composed only by inertia and gyroscopic matrices.

Let us rewrite Eq. (3) as a first order differential equation. Making $\dot{\alpha} = \beta$ and $\dot{z} = c$, where β represents the angular velocity of the disc and c represents the current in the electric circuit of the DC motor, it is possible to rewrite Eq. (3) as

$$M\dot{\mathbf{X}}(t) + G\mathbf{X}(t) = \mathbf{0}, \tag{4}$$

where $\mathbf{X} = \dot{\mathbf{Y}} = \begin{bmatrix} c \\ \beta \end{bmatrix}$. The initial condition turns into $\mathbf{X}(0) = \begin{bmatrix} c_0 \\ \beta_0 \end{bmatrix}$. Once the solution of the IVP involving Eq. (4) is obtained, it can be integrated to become the solution of the IVP involving Eq. (3). The constants of integration that will appear in this integration should be computed so that the initial condition $\mathbf{Y}(0)$ is satisfied.

Since l and j_m are considered to be non-zero, M is an invertible matrix. Thus, Eq. (4) can be rewritten as

$$\dot{\mathbf{X}}(t) = -M^{-1}\, G\, \mathbf{X}(t) = A\mathbf{X}(t), \tag{5}$$

where $A = -M^{-1}G$.

3 Solution of the system dynamics

We propose as solution to the dynamics of the simplest electromechanical system $\mathbf{X} = \mathbf{U}\, e^{\lambda\, t}$, where $\mathbf{U}$ is a non-zero constant vector and λ a scalar. Substituting the proposed general solution into the system dynamics, we get $(A - \lambda\, I)\mathbf{U} = \mathbf{0}$, which forms an eigenvalue problem.

3.1 Natural frequency and modes of the electromechanical system

Since $\mathbf{U} \neq \mathbf{0}$, the matrix $(A - \lambda\, I)$ is singular. Thus:

$$\det\Big(A - \lambda\, I\Big) = 0 \quad \Rightarrow \quad \lambda^2 + \frac{k_e^2}{l\, j_m} = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm \frac{k_e}{\sqrt{l\, j_m}}\, i\,, \tag{6}$$

where $i = \sqrt{-1}$. Substituting the two eigenvalues $\lambda_{1,2}$ into the eigenvalue problem, it is possible to write $(A - \lambda_1\, I)\mathbf{U}_1 = \mathbf{0}$ and $(A - \lambda_1\, I)\mathbf{U}_2 = \mathbf{0}$. For $\lambda_1 = \frac{k_e}{\sqrt{l\, j_m}}\, i$, the associated eigenvector is $\mathbf{U}_1 = \begin{bmatrix} i\, j_m/\sqrt{l\, j_m} \\ 1 \end{bmatrix}$. For $\lambda_2 = -\frac{k_e}{\sqrt{l\, j_m}}\, i$, the associated eigenvector is $\mathbf{U}_2 = \begin{bmatrix} -i\, j_m/\sqrt{l\, j_m} \\ 1 \end{bmatrix}$. The eigenvalues $\lambda_{1,2}$ give a natural frequency of the system $\omega_n = \frac{k_e}{\sqrt{l\, j_m}}$. The eigenvectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are modes. Observe that the natural frequency, ω_n, and the modes are hybrid. They involve mechanical and electromagnetic parameters. Since two pairs of eigenvalues and eigenvectors were found, the general solution will be a linear combination of the two found solutions $e^{\lambda_1\, t}\,\mathbf{U}_1$ and $e^{\lambda_2\, t}\,\mathbf{U}_2$. It can be written as:

$$\mathbf{X}(t) = a\, e^{\lambda_1\, t}\,\mathbf{U}_1 + b\, e^{\lambda_2\, t}\,\mathbf{U}_2 = \begin{bmatrix} \cos\left(\frac{k_e}{\sqrt{l\, j_m}} t\right) \frac{j_m}{\sqrt{l\, j_m}}\, h - \sin\left(\frac{k_e}{\sqrt{l\, j_m}} t\right) \frac{j_m}{\sqrt{l\, j_m}}\, d \\ \cos\left(\frac{k_e}{\sqrt{l\, j_m}} t\right) d + \sin\left(\frac{k_e}{\sqrt{l\, j_m}} t\right) h \end{bmatrix}, \tag{7}$$

where a and b are constants, $d = a + b$ and $h = i(a - b)$. The constants a and b are computed so that Eq. (7) satisfies the initial condition $\mathbf{X}(0)$. Thus,

$$h = \frac{\sqrt{l\, j_m}}{j_m} c_0, \qquad d = \beta_0\,. \tag{8}$$

Figures 2(a) and 2(b) show a solution of Eq. (5). The parameter values used in the construction of the graphs are listed in Table 1. The motor parameters were obtained from the specifications of the motor Maxon DC brushless number 411678. The initial conditions are $c_0 = 1.000$ Amp and $\beta_0 = 1.000$ rad/s.

Table 1: Parameter values.

$l = 1.880 \times 10^{-4}$ H	$k_e = 5.330 \times 10^{-2}$ V/(rad/s)
$j_m = 1.210 \times 10^{-4}$ kg m^2	

4 Asynchronous parametric excitation

An asynchronous parametric excitation is introduced in the system as shown Eq. (9),

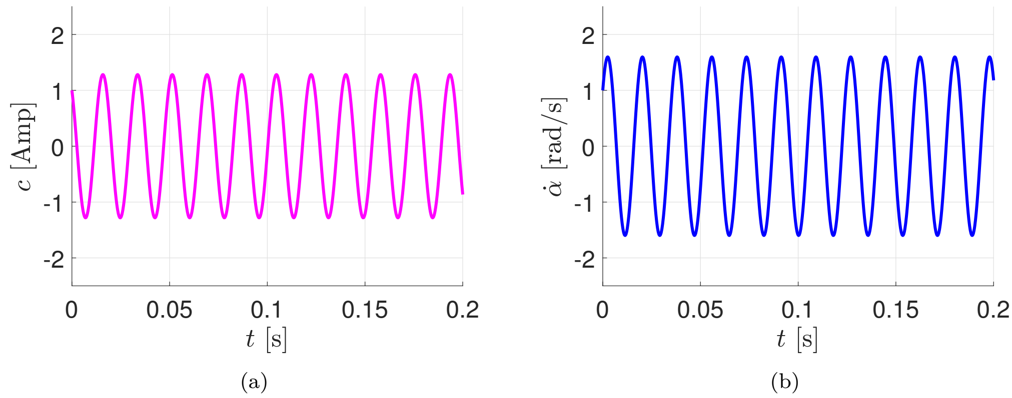

Figure 2: (a) Current and (b) angular velocity the disc.

$$\begin{bmatrix} l & 0 \\ 0 & j_m \end{bmatrix} \begin{bmatrix} \ddot{z}(t) \\ \ddot{\alpha}(t) \end{bmatrix} + k_e \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & \cos(\Omega t) \\ -\cos(\Omega t + \theta) & 0 \end{bmatrix} \right) \begin{bmatrix} \dot{z}(t) \\ \dot{\alpha}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (9)$$

The parametric excitation terms have an amplitude ϵ, frequency Ω and asynchrony with respect to each other θ.

Before starting the analysis of stability of the system with the parametric excitation, let us rewrite Eq. (9) as a first order differential equation. Making $\mathbf{X} = \dot{\mathbf{Y}} = \begin{bmatrix} c \\ \beta \end{bmatrix}$, it is possible to rewrite Eq. (9) as

$$\dot{\mathbf{X}}(t) = B(t)\mathbf{X}(t)\,, \quad (10)$$

where $B = -\begin{bmatrix} 1/l & 0 \\ 0 & 1/j_m \end{bmatrix} k_e \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \epsilon\cos(\Omega t) \\ -\epsilon\cos(\Omega t + \theta) & 0 \end{bmatrix} \right)$.

With a set of two linear independent solutions $(x_1,\, x_2)$ of Eq. (10), it is possible to construct a matrix called fundamental matrix as $\Phi(t) = [x_1\ x_2]$.

Floquet theorem affirms that the fundamental matrix $\Phi(t)$ with $\Phi(0) = I$ has a Floquet normal form $\Phi(t) = Q(t)e^{St}$ where $Q \in C^1(\mathbb{R})$ is a periodic invertible matrix with period T for all t and $S \in \mathbb{C}^{2\times 2}$ is a constant matrix given by $S = \frac{1}{T}\ln(\Phi(T))$ [1].

The solutions of $\dot{\mathbf{X}}(t) = B(t)\mathbf{X}(t)$ are products of periodic functions with e^{St} and stability is determined through the eigenvalues of S. However, in general, obtaining matrix S explicitly is not possible. Therefore, a numerical approximation of $\Phi(t)$ can be used to evaluate stability of the trivial solution of $\dot{\mathbf{X}}(t) = B(t)\mathbf{X}(t)$.

A numerical approximation of $\Phi(t)$ can be found by taking $\Phi(0) = I$ (identity matrix) to obtain the monodromy matrix:

$$R = e^{ST} = \Phi(T). \quad (11)$$

The eigenvalues of R, γ_1, γ_2 are known as Floquet multipliers. Therefore, the trivial solution of $\dot{\mathbf{X}}(t) = B(t)\mathbf{X}(t)$ can be classified as stable, asymptotically stable or unstable (in the sense of Lyapunov), using the following criterion given by Floquet's theory. If $|\gamma_n| \leq 1$ for $n = \{1,\, 2\}$, the trivial solution is stable. If $|\gamma_n| < 1$ for $n = \{1,\, 2\}$, is asymptotically stable. It is unstable otherwise. Another way to analyze the stability of the trivial solution $\dot{\mathbf{X}}(t) = B(t)\mathbf{X}(t)$ is using the Lyapunov characteristic exponents (LCEs) μ_n that are the real parts of the eigenvalues of B. They can also be obtained directly by the Floquet multipliers as $\mu_n = \frac{1}{T}\ln|\gamma_n|$.

The trivial solution is stable if $\mu_n \leq 0$ for $n = \{1, 2\}$; asymptotically stable if $\mu_n < 0$ for $n = \{1, 2\}$; unstable otherwise. Being $\Lambda = \max\{\mu_n\}$, it is sufficient that $\Lambda \leq 0$ for the trivial solution of $\dot{\mathbf{X}}(t) = B(t)\mathbf{X}(t)$ to be stable.

5 Stability maps

To analyze the influence of the parameters ϵ, Ω and θ in the classification of the stability of the trivial solution, stability maps were constructed for different combinations of these parameters. For each combination, the steps made in the numerical simulations are: 1) obtain an approximation of the fundamental matrix $\Phi(t)$ on the interval $[0,\ T]$ with initial conditions $\Phi(0) = I$; 2) compute the monodromy matrix $R = \Phi(T)$; 3) compute the eigenvalues of R, the Floquet multipliers; 4) obtain the LCEs and Λ in order to make the classification.

For each of the following values of θ, $\{0,\ 0.1,\ 0.3,\ 0.5,\ 0.7,\ 1\}\,\pi/2$, a stability map was constructed for values of Ω in the interval $[20,\ 1400]$ [rad/s] and ϵ in the interval $[0.1,\ 1]$. The maps are shown in Figure 3.

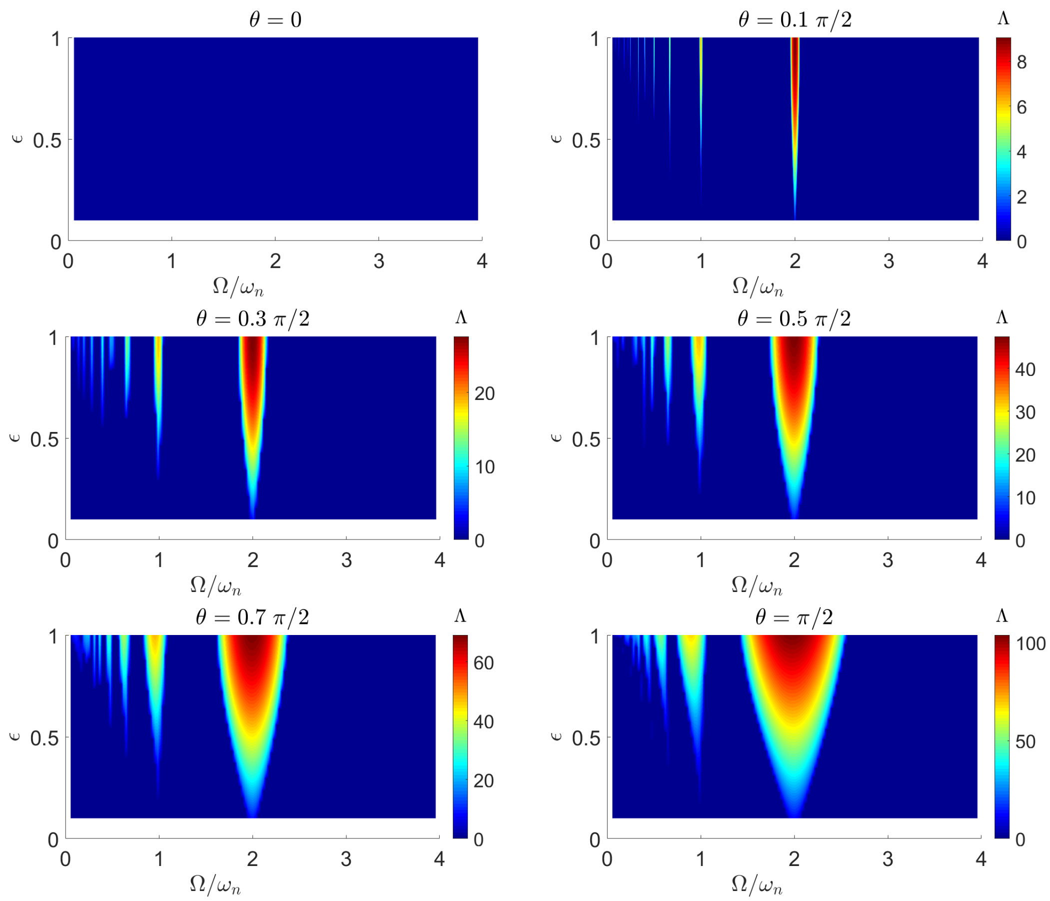

Figure 3: Stability maps for different values of θ.

It is possible to observe that for $\theta = 0$ there are no regions of instabilities. We will call this total stability. For $\theta > 0$, the instabilities appear and as θ increases, the regions of instability

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increase. Also the regions of instability in the parameter space of Ω and ϵ originate from the critical frequencies

$$\Omega\text{crit} = \frac{2\omega_n}{p}, \qquad p \in \mathbb{N}. \tag{12}$$

6 Conclusions

This work analyzed the influence of asynchronous parametric excitation in stability maps of the simplest electromechanical system. To construct the stability maps, numerical approximations of linear independent solutions of the system dynamics were computed and the stability was analyzed with the Lyapunov characteristic exponents. The focus of the work was to determine the influence of the phase between the parametric excitation terms in the stability of the trivial solution. Six stability maps were plotted with the aim of showing regions of stability and instability in function of parameters Ω and ϵ. It is possible to observe that for $\theta = 0$ there are no regions of instabilities, what we called total stability. For $\theta > 0$, the instabilities appear and as θ increases, the regions of instability increase.

Acknowledgments

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