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Fuzzy numerical solution to the Malthusian model via J_γ-interactive arithmetic

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Abstract. This work provides a numerical solution to the Malthusian model, considering the initial condition as a fuzzy value. The numerical solution is provided from Euler's method, in which the operations built into the method are adapted to fuzzy numbers. This numerical solution is compatible with the analytical solution obtained from the generalized Hukuhara derivative. An example is presented to illustrate the methodology.

Keywords. Fuzzy initial value problem, Generalized Hukuhara derivative, Euler's method, Malthus model, Fuzzy Interactivity

1 Introduction

The study of Fuzzy Differential Equations (FDE) are important to describe some phenomena that have some kind of uncertainty or imprecision, whether in the initial conditions/problem boundary or in the parameters involved. In this case, imprecision is modeled by fuzzy numbers, which are an extension of real numbers. For the study of FDEs, it is first necessary to take into account the type of derivative that rules the field. Our focus will be on the generalized Hukuhara derivative, which is the most used in the literature. To approximate the solution of a FDE according to the generalized Hukuhara derivative, Esmi et al. [6] proposed a numerical method based on a fuzzy arithmetic called J_γ-interactive arithmetic. The arithmetic was originally build to fuzzy numbers [7] and adapted to interval value in [6] in order to prove that the proposed numerical method is connected with the analytic solution of FDE via generalized Hukuhara derivative.

This work will use the theoretical results presented in [6] to provide a numerical solution for the Malthus population model. Malthus model considers no restriction of space and food of the population, so the population grows proportionally to itself. Here we will assume that the initial value of a population is uncertain and will be modeled by fuzzy numbers. The objective is to analyze how the uncertainty about the initial condition evolves over time, from the J_γ-interactive arithmetic.

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To illustrate the analysis and interpretation, we will perform a computer simulation presenting the numerical solution obtained considering two cases, the first is the case in which the generalized Hukuhara derivative of type (I), that is, expansive and the other is the case where the generalized Hukuhara derivative of type (II), that is, contractive [1].

The work is structured as follows: Section 2 provides preliminary results for a good understanding of the work. Section 3 presents the analytical solution via gH-derivative to the Malthusian model. Section 4 presents the numerical solution via interactive arithmetic and the comparison between them. Finally, Section 5 presents the final remarks of the work.

2 Mathematical Background

This subsection presents the fuzzy set theory and a mathematical background necessary for this paper.

First, we briefly present the numerical method that will be used in this work. Euler's method produces a numerical approximation to the solution of differential equations of the type $x' = f(t, x)$.

The method consists of, starting from an initial condition x_0, calculate:

$$x_{n+1} = x_n + hf(t_n, x_n)$$

where h is $h = t_{n+1} - t_n$ and x_n is the approximation of $x(t_n)$ [9].

A fuzzy subset A of a universe X is defined by a function called membership function φ_A that maps elements of X in [0,1]. This function can be interpreted as follows, as closer $\varphi_A(x)$ is to 1 the greater the membership of x in A, where $\varphi_A(x) = 1$ means total association with the set A. A fuzzy set can be also characterized with a class of real subsets called α-levels. For $0 < \alpha \leq 1$, the α-level of A is defined by the classical set $[A]^\alpha = \{u \in X : \varphi_A(u) \geq \alpha\}$. For the case where X is also a topological space, the 0-level of A is defined by $[A]^0 = cl\{u \in X : \varphi_A(u) > 0\}$, where clY denotes the closure of the subset $Y \subseteq X$.

As an extension of real numbers, a fuzzy number is defined from the following properties [4, 5]. A fuzzy subset A of $\mathbb{R}$ is a fuzzy number if it satisfies:

1: every α-level of A is a nonempty and close interval of $\mathbb{R}$;

2: $SuppA = \{u \in \mathbb{R} \mid \varphi_A(u) > 0\}$ is bounded.

The set of fuzzy numbers is denoted by $\mathbb{R}_\mathcal{F}$.

Since the α-levels of a fuzzy number A are intervals, we represent them by $[A]^\alpha = [\underline{a}_\alpha, \overline{a}_\alpha]$. The diameter of $[A]^\alpha$ is defined by $diam([A]^\alpha) = \overline{a}_\alpha - \underline{a}_\alpha$ and can be interpreted by the amount of uncertainty that it models. A translation of a fuzzy number A by the midpoint of $[A]^1$, i.e., $a = 0.5(\underline{a}_1 + \overline{a}_1)$, is the fuzzy number A^t with α-level given by $[A^t]^\alpha = [\underline{a}^t_\alpha, \overline{a}^t_\alpha]$. This translation leads to two interesting properties: $\underline{a}^t_\alpha \leq 0 \leq \overline{a}^t_\alpha$, for all $\alpha \in [0, 1]$ and $[A]^\alpha = [A^t]^\alpha + a$. Translations are important to construct the interactive arithmetic that we will presented further up.

The notion of interactivity arises whenever a joint possibility distribution (JPD) is given [8, 11]. A JPD for fuzzy numbers $A_1, \ldots, A_n$ is a fuzzy set J of $\mathbb{R}^n$ such that

$$\varphi_{A_i}(w) = \sup_{v \in \mathbb{R}^n, v_i = w} \varphi_J(v)$$

for all $w \in \mathbb{R}$ and $i = 1, \ldots, n$. In other words, if each fuzzy number A_i can be obtained by the projection in the i-th direction. In this sense, A_i is also called by marginal.

The sup-J extension of a function $m : \mathbb{R}^n \to \mathbb{R}$ at $A_1, \ldots, A_n$ is a fuzzy set $m_J(A_1, \ldots, A_n)$ of $\mathbb{R}$ with membership function

$$\varphi_{m_J(A_1,\ldots,A_n)}(u) = \sup_{(x_1,\ldots,x_n) \in m^{-1}(u)} \varphi_J(x_1, \ldots, x_n), \tag{1}$$

where $m^{-1}(u) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : m(x_1, \dots, x_n) = u\}$ [8].

Let J be a given JPD for fuzzy numbers $A_1, \dots, A_n$. If $J = J_\wedge$, we say that $A_1, \dots, A_n$ are non-interactive, otherwise, they are said to be interactive. In this case, if $J = J_\wedge$ then the sup-J extension is equal to the Zadeh's extension principle [11]. From this tool it is possible to define different arithmetics for fuzzy numbers. For example, the standard arithmetic is defined when $J = J_\wedge$ and here we will denote this operations by $+, -, \times, \div$, when $J \neq J_\wedge$ we denote the operations by $+_J, -_J, \times_J, \div_J$.

Unfortunately, the standard arithmetic lacks on good properties, such as the opposite element under the sum. In order to avoid these issues, several arithmetic operations were defined.

Given $A, B \in \mathbb{R}_\mathcal{F}$. If there exist a fuzzy number H such that

$$\begin{cases} (I)\ A = B + H \ or \\ (II)\ B = A - H \end{cases}, \tag{2}$$

then H is said to be the generalized Hukuhara difference (or, for short, the gH-difference) of A and B, which is denoted by $A -_{gH} B$.

Definition 2.1. *[2] A function $v : (a, b) \to \mathbb{R}_\mathcal{F}$ is said to be generalized Hukuhara differentiable (or, for short, gH-differentiable) at $t_0 \in (a, b)$ if there exists a fuzzy number $v'_{gH}(t_0) \in \mathbb{R}_\mathcal{F}$ such that*

$$v'_{gH}(t_0) = \lim_{h \to 0} \frac{v(t_0 + h) -_{gH} v(t_0)}{h},$$

where the limit is taken using the (Hausdorff) metric d_∞ [4].

Moreover, if the gH-difference $v(t_0 + h) -_{gH} v(t_0)$ satisfies the case (I) for all sufficient small $|h|$, then the function v is said to be gH-differentiable of type (I) or, simply, (I)-gH-differentiable at t_0. Similarly, if the gH-difference $v(t_0+h) -_{gH} v(t_0)$ satisfies the case (II) for all sufficient small $|h|$, then the function v is said to be gH-differentiable of type (II) or, simply, (II)-gH-differentiable at t_0.

This paper considers a family of JPDs given by (J_γ), $\gamma \in [0, 1]$, which was first presented by Esmi et al. [5]. The sup-J extension principle applied to J_γ gives raise to the following interactive sum.

Theorem 2.1. *[5] Let $A, B \in \mathbb{R}_{\mathcal{F}_C}$, whose α-levels are $[A]^\alpha = [a^-_\alpha, a^+_\alpha]$ and $[B]^\alpha = [b^-_\alpha, b^+_\alpha]$. Let $\overline{a} = \frac{\alpha^-_1 + \alpha^+_1}{2}$, $(a^{(\overline{a})})^-_\alpha = a^-_\alpha - \overline{a}$ and $(a^{(\overline{a})})^+_\alpha = a^+_\alpha - \overline{a}$. Hence for each $\gamma \in [0, 1]$, it follows that the α-levels of $A +_\gamma B$ are given by*

$$[A +_\gamma B]^\alpha = [c^-_\alpha, c^+_\alpha] + \{\overline{a} + \overline{b}\}$$

where,

$$c^-_\alpha = \inf_{\beta \geq \alpha} h^-_{(A+B)}(\beta, \gamma) \quad e \quad c^+_\alpha = \sup_{\beta \geq \alpha} h^+_{(A+B)}(\beta, \gamma),$$

with

$$\begin{aligned} h^-_{(A+B)}(\beta, \gamma) = \min\{ & (a^{(\overline{a})})^-_\beta + (b^{(\overline{b})})^+_\beta + \gamma((b^{(\overline{b})})^-_\beta - (b^{(\overline{b})})^+_\beta), \\ & (a^{(\overline{a})})^+_\beta + (b^{(\overline{b})})^-_\beta + \gamma((a^{(\overline{a})})^-_\beta - (a^{(\overline{a})})^+_\beta), \\ & \gamma((a^{(\overline{a})})^-_\beta + (b^{(\overline{b})})^-_\beta)\} \end{aligned}$$

and

$$\begin{aligned} h^+_{(A+B)}(\beta, \gamma) = \max\{ & (a^{(\overline{a})})^-_\beta + (b^{(\overline{b})})^+_\beta + \gamma((a^{(\overline{a})})^+_\beta - (a^{(\overline{a})})^-_\beta), \\ & (a^{(\overline{a})})^+_\beta + (b^{(\overline{b})})^-_\beta + \gamma((b^{(\overline{b})})^+_\beta - (b^{(\overline{b})})^-_\beta), \\ & \gamma((a^{(\overline{a})})^+_\beta + (b^{(\overline{b})})^+_\beta)\}. \end{aligned}$$

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The JPD J_1 is equivalent to the Cartesian product $A \times B$, consequently the sum $A +_{J_1} B$ is equivalent to the standard sum $A + B$. In this case, from the perspective of fuzzy set theory, the operands are called non-interactive. On the other hand, for $\gamma = 0$ we obtain the highest constrain in the Cartesian product. In this case the operands A and B have the highest "level" of interactivity, which is similar to the one of the cases of completely correlation [3].

The next two subsections present the analytical solution via generalized Hukuhara derivative and its numerical approximation via J_γ interactive sum to the Malthusian model.

3 Analytical solution to the Malthusian Model via gH-derivative

The Malthusian model is given by the following differential equation:

$$x'(t) = \lambda x(t). \tag{3}$$

Here we consider that the initial condition X_0 is given by a fuzzy number. Moreover, we consider that derivative of (3) is the gH-derivative. So, the Fuzzy Initial Value Problem (FIVP) associated with this problem is given by

$$\begin{cases} x'_{gH}(t) = \lambda x(t) \\ x(t_0) = X_0 \in \mathbb{R}_{\mathcal{F}_C} \end{cases}, \tag{4}$$

where $\lambda > 0$.

In order to solve the FDE given in (4) we can associated the fuzzy problem with a system of classical differential equations, such as

$$\begin{cases} (x'_{gH}(t))^-_\alpha = (\lambda x(t))^-_\alpha \\ x(t_0) = [(X_0)^-_\alpha, (X_0)^+_\alpha] \end{cases} \quad \text{and} \quad \begin{cases} (x'_{gH}(t))^+_\alpha = (\lambda x(t))^+_\alpha \\ x(t_0) = [(X_0)^-_\alpha, (X_0)^+_\alpha] \end{cases}, \tag{5}$$

if the gH-derivative is the type (I) and

$$\begin{cases} (x'_{gH}(t))^-_\alpha = (\lambda x(t))^+_\alpha \\ x(t_0) = [(X_0)^-_\alpha, (X_0)^+_\alpha] \end{cases} \quad \text{and} \quad \begin{cases} (x'_{gH}(t))^+_\alpha = (\lambda x(t))^-_\alpha \\ x(t_0) = [(X_0)^-_\alpha, (X_0)^+_\alpha] \end{cases}, \tag{6}$$

if the gH-derivative is the type (II).

If the gH-derivative is the type (I), then the solution is given by $x(t) = [(X_0)^-_\alpha, (X_0)^+_\alpha]e^{\lambda t}$ and if the gH-derivative is the type (II), then the solution is given by $x(t) = [c_1 e^{\lambda t} + c_2 e^{-\lambda t}, c_1 e^{\lambda t} - c_2 e^{-\lambda t}]$, where c_1 and c_2 are constants determined by the initial condition of the problem. Figures 1 and 2 depict these solutions. Note that the solution of type (I) is an expansive process, whereas the solution of type (II) is a contractive process. In all the next figures the gray-scale lines varying from white to black represent the α-levels of the analytical or numerical solution, where α varies from 0 to 1.

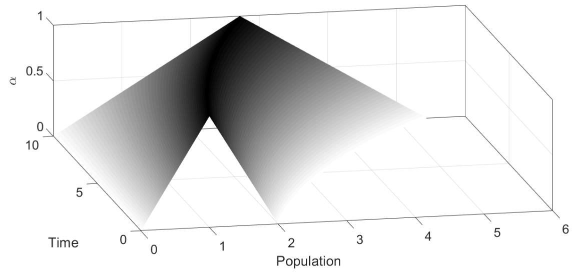

Figure 1: Analytical solution to the Malthusian model, considering the gH-derivative of type (I). The gray-scale lines varying from white to black represent the α-levels of the solution, where α varies from 0 to 1.

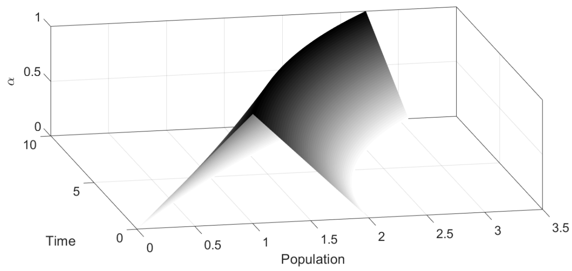

Figure 2: Analytical solution to the Malthusian model, considering the gH-derivative of type (II). The gray-scale lines varying from white to black represent the α-levels of the solution, where α varies from 0 to 1.

4 Fuzzy numerical solution to the Malthusian model via J_γ-interactive arithmetic

The numerical method considered here is given by the Euler's method, where the classical operations involved are adapted to fuzzy numbers [10]. In another words, the numerical solution via Euler's method is given by

$$X_{n+1} = X_n +_J h\lambda X_n, \tag{7}$$

where $+_J$ is an interactive sum.

According to Esmi et al. [6] the interactive arithmetic $+_{J_0}$ approximates the analytical solution via gH-derivative of type (I) for intervals values, whereas the arithmetic $+_{J_1}$ approximates the analytical solution via gH-derivative of type (II). Hence, we consider here two numerical methods:

$$X_{n+1} = X_n +_0 h\lambda X_n, \tag{8}$$

which we expect that approximates the fuzzy solution via gH-derivative of type (I) and

$$X_{n+1} = X_n +_1 h\lambda X_n, \tag{9}$$

which we expect that approximates the fuzzy solution via gH-derivative of type (II).

Figures 3 and 4 illustrate the numerical solution obtained for $\gamma = 1$ and $\gamma = 0$, respectively. Note that the numerical solution via $\gamma = 0$ is contractive and the numerical solution via $\gamma = 1$ is expansive, corroborating theoretical results [7].

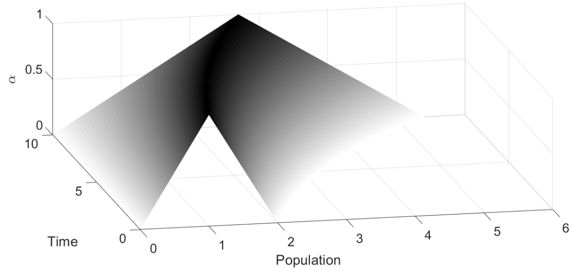

Figure 3: Fuzzy numerical solution to the Malthusian model considering the arithmetic $+_{J_1}$. The gray-scale lines varying from white to black represent the α-levels of the numerical solution, where α varies from 0 to 1.

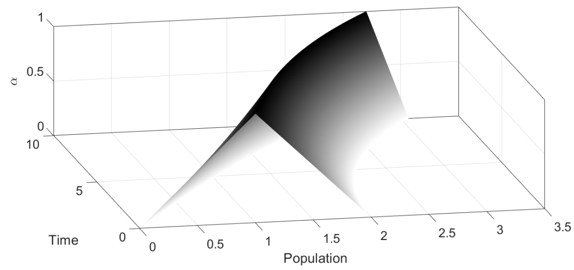

Figure 4: Fuzzy numerical solution to the Malthusian model considering the arithmetic $+_{J_0}$. The gray-scale lines varying from white to black represent the α-levels of the numerical solution, where α varies from 0 to 1.

The next figure presents the 0-level of the fuzzy analytical and numerical solutions in order to compare the results.

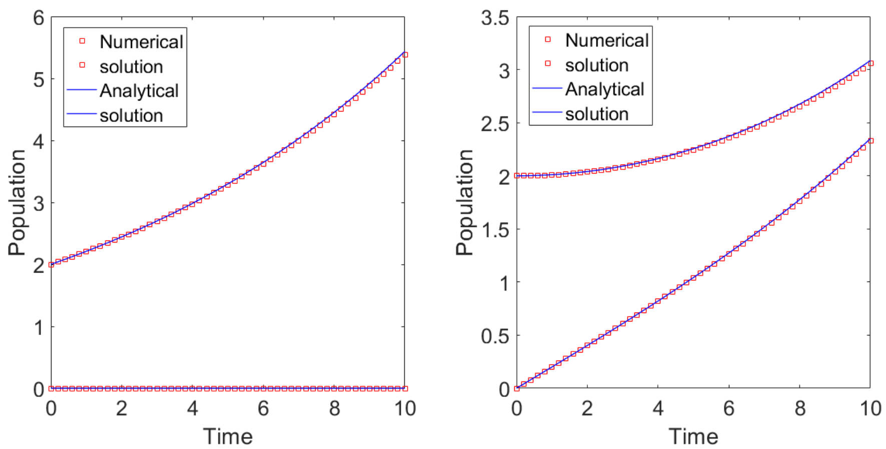

Figure 5: Comparison between the 0-levels of the analytical and numerical solutions to the Malthusian model. The left subfigure represents the expansive process, associated with the gH-derivative of type (I). The right subfigure represents the contractive process, associated with the gH-derivative of type (II). The blue lines represent the analytical solution and the red squares represent the numerical solution via interactive arithmetic.

5 Final Remarks

This work presents a numerical solution to the Malthusian model considering fuzzy numbers as initial value. The simulations presented here showed that the numerical solution via J_1 arithmetic approximates the analytical solution via gH-derivative of type (I), whereas the numerical solution via J_0 arithmetic approximates the analytical solution via gH-derivative of type (II), extending the results presented in [6].

As future work, we intend to prove that the numerical solution via J_1 approximates any analytical solution of gH-derivative of type (I) and the numerical solution via J_0 approximates any analytical solution of gH-derivative of type (II), for the fuzzy case.

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