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Recent Advances on a Stable Hybrid Finite Elements Methodology for Stokes Flow using H(div) Spaces

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The motion of an incompressible flow can be accurately described by the Navier-Stokes equations. When the viscous forces have a greater influence compared to the convective ones, eg: creeping flows, the governing equations can be simplified to the linear Stokes equations. Moreover, the flow is said to be in a steady state if the magnitude of the inertial forces is much smaller than the other terms, thus not varying its velocity in time.

Let $\Omega \in \mathbb{R}^d$ be a polygonal domain and its boundary $\partial\Omega \in \mathbb{R}^{d-1}$, with $d \in \{2,3\}$. Then, the Stokes problem consists of finding the velocity, $\underline{u}$, and the pressure, p, such that:

$$\text{div}\left(\underline{\underline{\sigma}}\right) = \underline{f} \text{ in } \Omega \tag{1}$$

$$\text{div}(\underline{u}) = 0 \text{ in } \Omega \tag{2}$$

$$\underline{u} = \underline{u}_D \text{ on } \partial\Omega_D \tag{3}$$

$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{\sigma}_N \text{ on } \partial\Omega_N \tag{4}$$

Eq. (1) describes the balance of momentum, in which $\underline{f}$ is an externally applied body force, and $\underline{\underline{\sigma}} = 2\mu D(\underline{u}) - p\underline{\underline{I}}$ is the stress tensor, where $D(\underline{u}) = \frac{1}{2}\left(\nabla\underline{u} + \nabla\underline{u}^T\right)$ is the rate of strain tensor, μ is the fluid viscosity, and $\underline{\underline{I}}$ is the identity matrix. Eq. (2) represents the conservation of mass.

The Dirichlet boundary condition is represented in (3), where $\underline{u}_D$ is the imposed velocity. Eq (4) represents the Neumann boundary condition, in which σ_N is the imposed surface traction, and $\underline{n}$ is the unit normal vector pointing outwards from the domain. The boundaries $\partial\Omega_D$ and $\partial\Omega_N$ represent the regions of the domain where the Dirichlet and Neumann boundary conditions are applied, respectively.

In many cases, the Stokes problem does not have an analytical solution, making it necessary to employ numeric procedures like the Finite Element Method (FEM). This research uses H(div)-conforming spaces for the velocity field, L^2-spaces for the pressure field, and a hybrid approach for the tangential traction between elements. The use of these spaces leads to a stable and locally conservative method. The procedure employed in this work to generate H(div)-conforming spaces is based on the multiplication of constant vectors and H^1-conforming basis functions [1].

By hybridizing the tangential stress, the continuity of this component is relaxed and weakly imposed. Therefore, Lagrange multipliers are employed over interface domains, Γ, to ensure the continuity of the tangential stress between elements. In this context, [2] proposes the eqs. (5), (6), and (7) as the weak formulation for the Stokes problems.

$$\int_\Omega \frac{\mu}{2} D(\underline{u}) \cdot D(\underline{w}_u)\, d\Omega - \int_\Omega p\text{div}(\underline{w}_u)\, d\Omega - \int_\Gamma w_{ut}\lambda_t\, d\Gamma = \int_{\partial\Omega_N} w_{un}\underline{\sigma}_n \cdot \underline{n}\, ds \quad \forall \underline{w}_u \in \mathbb{V}_u \tag{5}$$

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$$-\int_{\Omega} w_p \mathrm{div}(\underline{u})\, d\Omega = 0 \qquad \forall w_p \in \mathrm{L}^2(\Omega) \tag{6}$$

$$\int_{\Gamma} [\![u_t]\!] \eta_t \, d\Gamma - \int_{\partial\Omega_D} u\eta_t \, ds = -\int_{\partial\Omega_D} u_{Dt}\eta_t \, ds \quad \forall \eta_t \in \mathrm{H}^{-1/2}(\partial\Omega, \mathbb{R}^d) \tag{7}$$

in which $\underline{w}_u \in \mathbb{V}_u$, with $\mathbb{V}_u = \{\underline{w}_u | \underline{w}_u \in \mathrm{H}(\Omega, \mathrm{div}), w_{un} = 0 \text{ on } \partial\Omega_D\}$, is the velocity test function, $w_p \in \mathrm{L}^2(\Omega)$ is the pressure test function, λ_t is the Lagrange multiplier with physical meaning of tangential stress, $\eta_t \in \mathrm{H}^{-1/2}(\partial\Omega_e, \mathbb{R}^d)$ is the Lagrange multiplier test function, and $[\![\cdot]\!]$ is the jump operator between two elements, Ω_1 and Ω_2, defined by $[\![v]\!] = v|_{\Omega_1} - v|_{\Omega_2}$. When n and t appear as sub-indexes, they represent the normal and tangential components of velocity or tension, respectively.

To solve eqs. (5) - (7), the methodology used by [3] is employed. Figure 1 shows the solution of the Lid-Driven Cavity problem, a numerical test used to verify the robustness and stability of the proposed method.

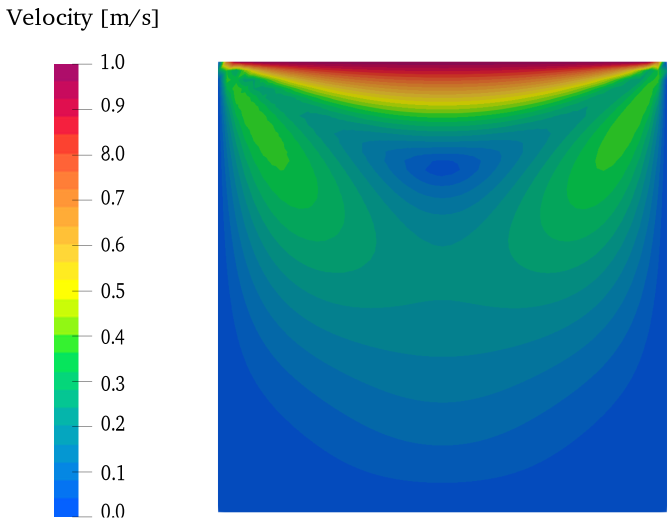

Figure 1: Velocity Profile for the Lid-Driven Cavity problem.

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