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Revisiting the Balanced Induced Subgraph Polytope

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A *signed graph* G is a triple $G = (V, E, s)$ where V is the vertex set, E is the set of undirected edges and $s : E \to \{+, -\}$ is a function that assigns a sign to each edge in E. Thus, E can be partitioned in two disjoint sets E^+ and E^-, such that $E^+ = \{e \in E : s(e) = +\}$ and $E^- = \{e \in E : s(e) = -\}$ and $E^+ \cup E^- = E$. We allow multiple edges in G as long as they have different signs. For convenience, denote by $E^+ \cap E^-$ the set of multiple edges in E[1].

Let $E(S)$, for $S \subset V$, be the set of edges with both endpoints in S. We say that a signed graph is balanced if V can be partitioned in U and $\bar{U}$ in such a way that $E^+ = E(U) \cup E(\bar{U})$ and E^- is the cut $[U, \bar{U}]$, that is all negative edges have one endpoint in U and the other one in $\bar{U}$. The Balanced Induced Subgraph Problem (BIS) consists in, given a signed graph $G = (V, E, s)$ with weights $w(v)$ for all $v \in V$, finding a balanced induced subgraph of G that maximizes the sum of weights of its vertices.

The notion of balance for signed graphs was firstly suggested in [2] to model certain concepts from problems in social psychology. In addition to being employed in the context of social networks, it has also been used to found problems from diverse areas such as finance [3, 4], document clustering [5] and biology [6]. In [2], the authors also provide some useful characterizations, such as the one that follows.

We say that a cycle $C \subset G$ is a *negative cycle* if it includes an odd number of negative edges, otherwise we call it a *positive cycle*. Denote $C^-(E)$ the set of all negative cycles of G. A signed graph is balanced if and only if it does not contain negative cycles This characterization inspires the following formulation for the BIS problem:

$$\max \sum_{u \in V} w(u) x_u \quad \text{s.a.} \sum_{u \in C} x_u \leq |C| - 1 \;\; \forall C \in C^-(E), \quad x \in \{0,1\}^{|V|}.$$

For every $W \subset V$, denote by $x^W \in \mathbb{R}^{|V|}$ the vector such that $x^W_u = 1$ if $u \in W$, and $x^W_u = 0$ otherwise. We study the polytope P_G defined as:

$$P_G = \text{conv}\{x^W \in \mathbb{R}^{|V|} : W \subset V,\ G[W] \text{ is balanced }\},$$

where $G[W] = (W, E(W), s)$. This is called the Balanced Induced Subgraph Polytope [7].

That being said, our first result is stated next;

Theorem. *Let $C \subset C^-(E)$. The negative cycle inequality $\sum_{u \in C} x_u \leq |C| - 1$ defines a facet of P_G if and only if $G[C]$ is chordless and for all $v \in V \setminus C$ there is $x \in C$ such that $G[C \setminus \{x\} \cup \{v\}]$ is balanced.*

It should be mentioned that necessary and sufficient conditions for the negative cycle inequality to be facet-defining were previously presented in [1, 7]. However, we could prove that they are only necessary conditions.

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Now, consider the following statement equivalent to BIS: starting from V, we want to remove the least amount of vertices to make G balanced. With this in mind, we propose a set covering formulation for BIS. For all $u \in V$, define $y_u \in \{0,1\}^{|V|}$ such that $y_u = 1$ if we remove u from G, and $y_u = 0$ otherwise. We obtain

$$\min \sum_{u \in V} w(u) y_u \quad \text{s.a.} \sum_{u \in C} y_u \geq 1 \ \forall C \in C^-(E), \quad y \in \{0,1\}^{|V|}.$$

Similarly, for all $S \in V$, denote by $y^S \in \mathbb{R}^{|V|}$ the vector such that $y_u^S = 1$ if $u \in S$, and $y_u^S = 0$ otherwise. Let $\hat{P}_G = \text{conv}\{y^S \in \mathbb{R} : S \subset V,\ G[V \setminus S] \text{ is balanced}\}$.

Since $y_u = 1 - x_u$ for all $u \in V$, there exists an affine transformation from P_G to $\hat{P}_G$, and vice versa, hence each facet of P_G has a corresponding facet in $\hat{P}_G$.

For example, the following statement holds:

Corollary. *Let $C \subset C^-(E)$. Then inequality $\sum_{u \in C} y_u \geq 1$ defines a facet of $\hat{P}_G$ if and only if $G[C]$ is chordless and for all $v \in V \setminus C$ there is $x \in C$ such that $G[C \setminus \{x\} \cup \{v\}]$ is balanced.*

There are many known facet defining inequalities for set covering problems. For instance, [8–10] derived all facets with coefficients in $\{0, 1, 2, 3\}$. We are currently studying the relation between these facets and facets of P_G, as well as which combinatorial properties of G relate to each known facet for the set covering problem.

We expect to obtain more facet defining inequalities for P_G and $\hat{P}_G$ before the conference date, and also find other interesting structures of signed graphs that relate to the facets of these two polytopes.

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