Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 10, n. 1, 2023.

Trabalho apresentado no XLII CNMAC, Universidade Federal de Mato Grosso do Sul - Bonito - MS, 2023

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Some results concerning global existence results for solutions of general conservative advection-diffusion equations in $\mathbb{R}$

Patrícia L. Guidolin[1], Linéia Schütz[2], Juliana S. Ziebell[3], Janaína P. Zíngano[4]
UFRGS, Porto Alegre, ,RS

The main idea of this work is to present some results of [1] about an asymptotic behavior of non-negative bounded solutions to the initial value problem

$$\begin{aligned} &u_t + (b(x,t)u^{k+1})_x = \mu(t)u_{xx} \quad \forall x \in \mathbb{R},\ t > 0, \\ &u(\cdot,0) = u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \end{aligned} \tag{1}$$

for arbitrary continuously differentiable advection fields b satisfying

$$b(\cdot,t) \in L^\infty_{loc}([0,\infty), L^\infty(\mathbb{R})) \ \ \forall x \in \mathbb{R},\ t \geq 0; \tag{2}$$

$$\left| \frac{\partial b(x,t)}{\partial x} \right| < B(t) \ \ \forall x \in \mathbb{R},\ t \geq 0, \tag{3}$$

for some $B \in C^0([0,\infty))$, $\mu(t) \in C^0([0,\infty))$ positive and $k \geq 0$ constant.

Here, by a (bounded) solution of problem (1) in some time interval $[0, T_*)$, we mean any function $u \in C^0([0,T_*), L^1$
$(\mathbb{R})) \cap L^\infty_{loc}([0,T_*), L^\infty(\mathbb{R}))$ satisfying the equation of problem (1) and converging to u_0 in the sense L^1_{loc} when $t \to 0$. For results on local (in time) existence of such solutions see [2] and [3], Ch. 7. It is also known that the solutions are unique, as can be shown using comparison principles, (see [4], Theorem 2.1).

The aim of the article [1] is to investigate for which values of k it is possible to guarantee the global existence of solutions to the problem (1). For this purpose, knowing that solutions can be extended to broader intervals of existence as long as they remain limited, it is important to examine the behavior of high norms in the $[0,T_*)$, in particular, the L^∞-norm.

In this way, we are going to present the main ideas of the proof of the following theorem:

Theorem 0.1. *Let* $\bar{q} \geq 1$, $0 \leq t_0 < t < T_*$ *and* $0 \leq k < 2\bar{q}$ *in* (1). *Then*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq U_\infty(t_0;t) \leq \bar{q}^{\frac{1}{2\bar{q}-k}} \max\left\{ \|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})}; B_\mu(t_0;t)^{\frac{1}{2\bar{q}-k}} U_{\bar{q}}(t_0;t)^{\frac{2\bar{q}}{2\bar{q}-k}} \right\}, \tag{4}$$

where $U_{\bar{q}}(t_0;t)$, $B_\mu(t_0;t)$ *is defined by*

$$U_q(t_0;t) := \sup\{\|u(\cdot,\tau)\|_{L^q(\mathbb{R})}; t_0 \leq \tau \leq t\}, \tag{5}$$

[1] patiguidolin@gmail.com
[2] lineia.schutz@ufrgs.br
[3] julianaziebell@ufrgs.br
[4] janaina.zingano@ufrgs.br

$$B_\mu(t_0;t) := \sup\left\{\frac{B(\tau)}{\mu(\tau)}; t_0 \le \tau \le t\right\}, \tag{6}$$

and

$$J(q) \equiv J(q,k,K) := K^{\frac{2}{q}}\left(\frac{q^2}{4(q+k)}\right)^{\frac{1}{2(q-k)}}, \tag{7}$$

In particular, if we take $t_0 = 0$ e $\bar{q} = 1$ in Theorem 0.1, by mass conservation, we have the next corollary.

Corollary 0.1. *Let $u \in C^0([0,T_*), L^1(\mathbb{R})) \cap L^\infty_{loc}([0,T_*), L^\infty(\mathbb{R}))$ be any given non-negative solution to problem (1) under hypotheses (2), (3). Then, $u(\cdot,t)$ is globally defined ($T_* = \infty$) and satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \max\left\{\|u_0\|_{L^\infty(\mathbb{R})}; B_\mu(0;t)^{\frac{1}{2-k}}\|u_0\|_{L^1(\mathbb{R})}^{\frac{2}{2-k}}\right\}, \tag{8}$$

for all $t > 0$ and $0 \le k < 2$.

With this results, we are able to derive an estimate for the limit $\limsup_{t\to\infty}\|u(\cdot,t)\|_{L^\infty(\smile)}$ in order to show that problem (1) has global solution for some values of k. That is, we are able to prove the following theorem:

Theorem 0.2. *Let $\bar{q} \ge 1$, $0 \le k < 2\bar{q}$ and $\mathcal{B}_\mu$, $\mathcal{U}_{\bar{q}}$ as defined above,*

$$\limsup_{t\to\infty}\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \le C_\infty^{\frac{1}{\bar{q}}}\,\bar{q}^{\frac{1}{2\bar{q}-k}}\,\mathcal{B}_\mu^{\frac{1}{2\bar{q}-k}}\,\mathcal{U}_{\bar{q}}^{\frac{2\bar{q}}{2\bar{q}-k}}, \tag{9}$$

where C_∞ is a constant and

$$\mathcal{B}_\mu := \lim_{n\to\infty} B_\mu(t_0^n;\infty) = \limsup_{t\to\infty}\frac{B(t)}{\mu(t)} \;\;,\;\; \mathcal{U}_{\bar{q}} := \lim_{n\to\infty} U_{\bar{q}}(t_0^n;\infty) = \limsup_{t\to\infty}\|u(\cdot,t)\|_{L^{\bar{q}}}; \tag{10}$$

In particular, in the important case $\bar{q} = 1$ considered above, from mass conservation, the following corollary is obtained.

Corollary 0.2. *Let $0 \le k < 2$ and $\mathcal{B}_\mu$ as defined above,*

$$\limsup \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \le C_\infty \mathcal{B}_\mu^{\frac{1}{2-k}}\|u_0\|_{L^1(\mathbb{R})}^{\frac{2}{2-k}}, \tag{11}$$

where C_∞ is a constant.

In this case, $u(\cdot,t)$ has a uniform limitation for all time $t > 0$.

Referências

[1] P.L. Guidolin, L. Schütz, J.S. Ziebell e J.P. Zingano. “Global existence results for solutions of general conservative advection-diffusion equations in $\mathbb{R}$”. Em: **Journal of Mathematical Analysis and Applications** 515 (2022), p. 126361. DOI: 10.1016/j.jmaa.2022.126361.

[2] O. A. Ladyzhenskaya, V. A. Solonnikov e N. N. Uralceva. **Linear and Quasilinear Equations of Parabolic Type**. Vol. 23. Providence: American Mathematial Society, 1968. ISBN: 0821815733.

[3] D. Serre. **Systems of Conservation Laws**. Vol. 1. Cambridge: Cambridge University Press, 1999. ISBN: 0521582334.

[4] N.L. Diehl, L. Fabris e P. R. Zingano. “Comparison results for smooth solutions of quasilinear parabolic equations”. Em: **Adv. Diff. Eqs. Control Proc.** 14 (2014), pp. 11–22.