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Navier Stokes modeled with MFEM

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Finite Element methods can be used to model fluid dynamics problems using Navier Stokes equation in several different ways but we are interested in compare two approaches for a physical application. The challenge in using a complex geometry immersed in a fluid flow is how to construct a mesh that can be adapted near the object's surface with possible irregular edges or cramped sides. It is possible to construct unstructured meshes in order to do that but it is also possible to use a penalization term (or forcing term $\mathbf{f}$) within the Navier Stokes momentum equation (1) and use a Cartesian mesh as our domain Ω which is computationally cheaper.

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \Delta \mathbf{u} = \mathbf{f}, \quad \text{in } \Omega. \tag{1}$$

MFEM library has a solver for the incompressible Navier Stokes equation which takes equation (1) and uses this in a dimensionless form yielding,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \frac{1}{\mathrm{Re}} \Delta \mathbf{u} = \mathbf{f}, \quad \text{in } \Omega. \tag{2}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega. \tag{3}$$

Where Re is the Reynolds number [1]. The solver necessarily works with structured meshes, however, we can store the geometrical information of a solid's volume $\Omega_0 \subset \Omega$ into the forcing term by defining a characteristic function $\chi(\mathbf{x}, t)$ depending on whether a point $\mathbf{x} \in \Omega$ belongs to the solid's volume Ω_0 or not avoiding having to work with complicated unstructured meshes. The advantage of this method is that χ can have an explicit dependence on time, which allows us to model moving solids without generating a new mesh at every time step. On the surface of the closed volume the velocity must satisfy the no-slip condition $\mathbf{u} = \mathbf{u}_0$ where $\mathbf{u}_0$ is the velocity of the solid itself. Therefore, a suitable way to define the penalization term is

$$\mathbf{f} = \frac{1}{\eta} \chi(\mathbf{x}, t)(\mathbf{u} - \mathbf{u}_0), \tag{4}$$

where $\eta = K/\nu > 0$, the scalar K is the measure of the flow conductance through the solid's volume seen as a porous medium and it is called the permeability[4], it does not depend on the nature of the fluid instead it depends on the geometry of the porous medium and has dimensions of m^2 [2]. Permeability can also be seen as $K = \mathrm{Da} \cdot d^2$ where Da is a dimensionless number known as the Darcy number and d is a characteristic length of the medium. In the limit $\eta \longrightarrow 0$, the fluid satisfies the Navier–Stokes equations with $\mathbf{f} = 0$ in $\Omega \backslash \Omega_0$ just as if we were not define the fictitious

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[4]In general, the permeability is a second-order tensor but assuming an isotropic medium, it reduces to a scalar.

domain Ω_0, and satisfies the Darcy's law in Ω_0 [3].

Our mathematical treatment is based on addressing the steady and linear case known as the Stokes problem. We can see the Stokes problem as a saddle point problem and prove that it is well-posed by taking into account that from the point of view of the calculus of variations, the Stokes problem corresponds to the minimization of a functional. It is also possible to prove that the Stokes problem with the penalized term:

$$\nabla p - \nu \Delta \mathbf{u} + \frac{\nu}{K}\mathbf{u} = \mathbf{f}, \quad \text{in } \Omega, \tag{5}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \tag{6}$$

is also well-posed [4]. We are going to quantify how much this penalization method is actually computational better than the conventional method in which we define meshes considering the solid's walls. As well, we see how accurate our solution is, in relation to the exact analytical solution for a Stokes problem. For this purpose, we use an axisymmetrical flow example [5] in hydrodynamics with exact solution: a ball of radius a, in uniform translation U as shows figure 1.

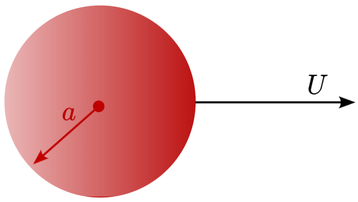

Figure 1: Translating sphere.

This allows us to test the precision of the numerical method and to find an approximation sufficiently near to the theory. After being sure that our method is sufficiently accurate, we enhance the complexity of the physical system by incorporating rotation within the penalized term and adding body forces like gravity into a new forcing term. Finally, we use the MFEM Navier Stokes solver in order to see a time evolution of the physical phenomenon for the flying sphere.

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