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# On the Spread of the Wildfire with a Time-space-varying Wind

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**Abstract**. In this paper, we present the model of wildfire propagation in flat terrain under the influence of a time-space-varying wind. We outline the methodology through systematic steps, facilitating the model development. Using the MATLAB environment, we simulate multiple experimental wildfire scenarios to demonstrate the practical application of our results in addressing real-world problems. Our results offer insights into determining optimal locations for constructing barriers to mitigate fire spread, known as the blocking problem. Implementation of our results can lead to reductions in the area affected by fire and decreased operational time and cost.

**Keywords**. Randers metric, firefront, time-space-varying wind, strategic path, blocking problem.

## 1 Introduction

The paper underscores the importance of mathematical approaches in systematically and accurately addressing wildfire propagation problems. Finsler geometry emerges as a robust tool for modeling wildfire waves [5–7, 9, 10, 12]. Using ellipses to model wildfire propagation via Huygens' principle is widely used [1, 14]. While simulators like FARSITE [8] utilize ellipses to determine firefront positions, their reliance on ellipses as spherical firefronts may not always reflect real-world conditions accurately [13], due to the complexity of the conditions. To address these limitations, recent research has explored Finsler geometry's techniques, demonstrating the validity of Huygens' principle and providing wildfire propagation models [7, 10, 12, 13]. In [10] and [13], the authors studied fire propagation by respectively applying the cone structure and the so-called frozen metric. Detailed studies on propagation under time-space-varying winds remain scarce. To address this gap, this paper focuses on providing solutions to the following real-world problems:

**Problem.** *There is a fire spreading in an area in which conditions on the terrain could change smoothly across the space. A time-space varying wind blows, although it remains time-independent within intervals of time. What would be the shape of propagation? How should the operational team allocate firefighters and equipment to manage the operation efficiently? Where are the optimal locations to construct barriers to block the fire from advancing in a particular direction?*

To address this problem, this paper presents a step-by-step approach to deriving equations for firefronts, fire rays, and strategic paths under time-space-varying winds. Also, the paper discusses the practical applications of strategic paths in risk management and resource allocation, that is strategic paths applications in blocking problems [4].

## 2 Basic Concepts Involved in Solving the Problem

Let $M$ be an open subset of $\mathbb{R}^2$, $p = (x, y)$ each point of $M$ and $T_pM$ vectors tangent at $p$. We represent each vector $V$ in $T_pM$ as $V = (u, v)$, using the standard basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ of $\mathbb{R}^2$. A

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*Riemannian metric* [11] on $M$ is a smooth function $h$ that to each $p \in M$, it assigns a positive-definite inner product $h_p : T_pM \times T_pM \to \mathbb{R}$. The length of a vector $V$ with respect to $h$ is $||V||_h = \sqrt{h(V,V)}$. Given a Riemannian metric $h$ and a vector field $W$, satisfying $h(W,W) < 1$, the function

$$F(V) = \frac{\sqrt{h^2(W,V) + \lambda h(V,V)}}{\lambda} - \frac{h(W,V)}{\lambda}, \tag{1}$$

where $\lambda = 1 - h(W,W)$ is called a *Randers metric* [2]. The length of each vector $V$ concerning $F$ is $F(V)$. A smooth curve $\gamma : [0,1] \longrightarrow M$ in the Randers space $(M,F)$ is called a *geodesic* if it is locally the shortest time path connecting each two nearby points on $\gamma([0,1])$. The length of $\gamma$ is $L[\gamma] := \int_0^1 F(\gamma'(t))dt$. The Riemannian geodesic and length of a curve in Riemannian space are defined likewise. Given a smooth vector field $W$, the flow of $W$ is the smooth map $\varphi : (-\epsilon,\epsilon) \times M \longrightarrow M$ such that, for each $t$, $\varphi_t : M \longrightarrow M$ is invertible and $\varphi_0$ is the identity map, and for every $p \in M$, $\varphi^p(t) := \varphi(t,p)$ satisfies $\frac{d\varphi^p}{dt}(t)|_{t=0} = W(p)$. Given a Riemannian metric, the vector field $W$ is called a Killing vector field if $\varphi_t$ preserves the metric, that is $\varphi_t^* h = h$ [11].

A firefront at each time refers to the perimeter of the burnt area. If the fire originates from a single point, the firefront after one unit of time is a spherical firefront. A fire ray denotes the path of a fire particle. A strategic path represents a path through which the fire engulfs more area or reaches a region that requires protection from the fire. The following results, sourced from reference [5], are being adjusted to suit the framework of this paper.

**Theorem 2.1.** *[5] Here, a wildfire spreads in $M$ and is influenced by the wind $W$, with $A$ representing the firefront at time 0. The factors impacting wildfire dynamics vary smoothly. Then:*

*(i) The spherical firefront of radius $\tau$ at each point $p \in M$ is $W$-translation of the ellipse*

$$Q(u,v) = \left(\frac{u\cos\theta - v\sin\theta}{a}\right)^2 + \left(\frac{u\sin\theta + v\cos\theta}{b}\right)^2 = 1, \tag{2}$$

*in which $a, b$, and $\theta$ are smooth functions and determined by experimental data.*

*(ii) The fire rays are Finsler geodesic which are $F$-unitary and orthogonal to $A$, where $F$ is given by Eq. (1) and $h = \frac{1}{2}\mathtt{Hess}\, Q$.*

*(iii) The firefront at time $\tau$ is $\{\gamma(\tau) : \gamma(t)$ are fire rays starting from $A\}$.*

*(iv) If all points of $M$ have equal priority for fire protection, at each time $\tau$, the strategic path is the fire ray $\gamma(t)$ for which $||\gamma(\tau) - \gamma(0)||_{Euc}$ is maximized.*

*(v) For each area $B$, the strategic path that intersects $B$ is the fire ray $\gamma(t)$ for which $\gamma(\tau) = q$. Here, $\tau$ is the time when the firefront first intersects $B$ and $q$ is the point of intersection.*

**Remark 2.1.** *In Theorem 2.1, by the $W$-translation of the ellipse we mean shifting every ellipse point by $W$. Formally, if you have an ellipse defined by the equation $\frac{(u-h)^2}{a^2} + \frac{(v-k)^2}{b^2} = 1$, where $(h,k)$ is the center of the ellipse and $a$ and $b$ are the semi-major and semi-minor axes respectively, then translating this ellipse by a vector $W = (W_1, W_2)$ results in a new ellipse with the same shape and orientation, but with a new center at $(h + W_1, k + W_2)$.*

If the wind is a Killing vector field, we have a simplified version of Theorem 2.1 as follows.

**Corollary 2.1.** *[5] Assume that a wildfire spreads in $M$, the Killing wind $W$ blows, $A$ is the firefront at time 0 and the factors influencing the wildfire dynamics change smoothly. Then:*

*(i) The fire ray from each $p \in A$ is $\gamma(t) := \varphi(t, \alpha(t))$, where $\varphi$ is the flow of $W$ and $\alpha$ is the Riemannain geodesic with initial condition $\alpha(0) = p$, $||\alpha'(0)||_h = 1$, and $d\varphi_p \alpha'(0) \perp_h A$.*

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*(ii) The firefront at time $\tau$ is $\{\varphi(\tau, \alpha(\tau)) : \varphi(t, \alpha(t))$ are fire rays emanating from $A\}$.*

*(iii) If all points of $M$ have equal priority for fire protection, at each time $\tau$, the strategic path is the fire ray for which $||\alpha(\tau) - \alpha(0)||_{Euc}$ is maximized.*

*(iv) At each area $B$, the path toward $B$ is $\varphi(t, \alpha(t))$ such that $\varphi(\tau, \alpha(\tau)) = q$. Here, $\tau$ is the time when the fire first intersects $B$ and $q$ is the point of arrival.*

# 3 Steps to Solve the Problem

Here, a wildfire propagates in a flat land $M$, under a smooth distribution of conditions across $M$ and a time-space-varying wind $W(t, p)$, where $p \in M$, $t \in [0, T]$, and $T$ is the total propagation time. A time-varying wind means that the wind changes with time; however, it remains time-independent as $W_i(t_i, p)$ during each time interval $[t_i, t_{i+1}]$, where $0 = t_0 < t_1 < \cdots < t_n = T$ forms a partition of $[0, T]$. Considering this kind of wind is typical as, in reality, the wind remains unchanged for about an hour before altering its direction or speed. We assume that the firefront at time $t_i$, denoted as $A_i$, is given, and we aim to obtain the fire rays, firefronts, and strategic paths equations for $[t_i, t_{i+1}]$.

Given a fire propagation in flat land, there is an associated ellipse given by Eq. (2). Through some straightforward calculations, from the ellipse, one obtains the Riemannian metric as

$$h_{i(x,y)} = \big(\frac{1}{b_i^2} - \frac{1}{a_i^2}\big)\big(\cos^2\theta_i(du^2 - dv^2) - \sin 2\theta_i du dv\big) + \big(\frac{du^2}{b_i^2} + \frac{dv^2}{a_i^2}\big), \tag{3}$$

and by replacing Eq. (3) in Eq. (1), we find the Randers metric. Observe that in Eq. (3), once we substitute a point $p = (x, y)$, we will have an inner product in the tangent space $T_pM$. Now, we have all the necessary conditions to solve the Problem addressed in the Introduction section. Through the next sections, we provide the steps for solving problem 1 for cases of stable (uniform) and variable terrain conditions, while assuming that $||W||_h < 1$.

## 3.1 Stable Terrain Conditions

Here, the fire spreads in a flat terrain under uniform conditions. The wind, which varies with time but remains constant as $W_i$ during the time interval $[t_i, t_{i+1}]$, blows.

Step 1. Determine constant values $a_i, b_i$, and $\theta_i$ in Eq. (2), using experimental data;

Step 2. Write the metric $h_i$ as given by Eq. (3);

Step 3. Find the fire rays as $\gamma_i(t) = p_i + tV_i$, $t \in [0, t_{i+1} - t_i]$, where $p_i$ belongs to $A_i$ and $V_i$ is a vector such that $V_i - W_i$ is an $h_i$-unitary vector and $h_i$-orthogonal to $A$;

Step 4. For each time $\tau \in [t_i, t_{i+1}]$, write the forefront as the set $\{\gamma_i(\tau)\}$, where $\gamma_i(t)$ represent the fire rays obtained in the step 3;

Step 5. Find the strategic path as the fire ray $\gamma_i(t) = p_i + tV_i$ such that:

Case i. $||V_i||_{\text{Euc}}$ is maximum among all rays starting from $A_i$, for the strategic path that maximizes the burnt area until time $\tau$;

Case ii. $\gamma_i(\tau) = q \in B$, in which $\tau$ is the first time when the fire reaches $B$, for the strategic path reaching a specific area $B$.

 

The constancy of the wind implies that its flow consists of a family of straight lines. The stability of the conditions implies that the Riemannian metric components are constant functions and, therefore, the Riemannian geodesics are straight lines. By performing straightforward calculations based on these facts and applying Corollary 2.1, one can provide Steps 1-5 below to solve problem 1 and present the equations for firefronts, fire rays, and strategic paths during the interval $[t_i, t_{i+1}]$.

## 3.2 Non-stable Terrain Conditions

Here, a wildfire spreads under the non-stable terrain conditions and the wind $W_i(.) := W(t_i, .)$ blows during the time $[t_i, t_{i+1}]$. We consider two cases for the wind: (1) $W_i$ is a Killing vector field, and (2) $W_i$ is a non-Killing vector field. It is well-known that a vector field on the Riemannian space $(M, h)$ is Killing if [11]

$$\sum_{k=1}^{2}(W^k \frac{\partial h_{rj}}{\partial x^k} + h_{kj}\frac{\partial W^k}{\partial x^r} + h_{rk}\frac{\partial W^k}{\partial x^j}) = 0, \qquad r, j = 1, 2. \tag{4}$$

### 3.2.1 A Killing Wind

The following steps outline the process of finding the propagation model for the time interval $[t_i, t_{i+1}]$ under the presence of the wind $W_i$ that satisfies Eq. (4).

Step 1. Determine the smooth functions $a_i, b_i$, and $\theta_i$ in Eq. (2), using experimental data;

Step 2. Write the metric $h_i$ by Eq. (3);

Step 3. For every point $p$ of $A_i$, write the the ray from $p$ as $\varphi_i(t, \alpha_i(t)),\ t \in [0, t_{i+1} - t_i]$, where $\varphi_i$ is the flow of $W_i$ and $\alpha_i(t) = (x_i(t), y_i(t))$ is the solution of

$$\frac{d^2x^r}{dt^2} + \frac{1}{2}\sum_{l,k,j=1}^{2} h^{rl}(\frac{\partial h_{lk}}{\partial x^j} + \frac{\partial h_{lj}}{\partial x^k} - \frac{\partial h_{kj}}{\partial x^l})\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \qquad r = 1, 2, \tag{5}$$

such that $\alpha_i(0) = p$, $\alpha_i'(0)$ is $h_i$-unitary and $h_i$-orthogonal to $A_i$;

Step 4. At time $\tau \in [t_i, t_{i+1}]$, write the forefront as the set $\{\varphi_i(\tau, \alpha_i(\tau))\}$, where $\varphi_i(t, \alpha_i(t))$ are obtained in step 3;

Step 5. Find the strategic path as the fire ray $\varphi_i(t, \alpha_i(t))$ such that:

Case i. $||\varphi_i(\tau, \alpha_i(\tau)) - \alpha_i(0)||_{\text{Euc}}$ is the maximum among all the fire rays initiating from $A_i$, for the strategic path that maximizes the burnt area until time $\tau$;

Case ii. $\varphi_i(\tau, \alpha_i(\tau)) = q \in B$, in which $\tau$ is the first time when the fire reaches $B$, for the strategic path reaching a specific area $B$.

We provide steps 1-5, by doing straightforward calculations and applying Corollary 2.1, considering the metric preserving properties of the flow of a Killing vector field.

### 3.2.2 A Non-Killing Wind

We apply Theorem 2.1 to outline the steps for presenting the propagation model for the wind that does not satisfy Eq. (4). Steps 1 and 2 mirror those in 3.2.1, and we skip them, proceeding directly to step 3.

Step 3. For every point $p$ belonging to $A_i$, the fire ray igniting from $p$ is $\gamma_i(t) = (x_i(t), y_i(t))$, $t \in [0, t_{i+1} - t_i]$, where $\gamma_i(t) = (x_i(t), y_i(t))$ is the solution of [15]

$$\frac{d^2x^r}{dt^2} + \frac{1}{2}\sum_{l,k,j=1}^{2} g^{rl}\left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^l}\right)\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \qquad r = 1, 2, \tag{6}$$

where $[g_{rj}] = \frac{1}{2}\frac{\partial^2 F^2}{\partial v_r \partial v_j}$, such that $\gamma_i(0) = p$ and $\gamma_i'(0)$ is $F_i$-unitary and $F_i$-orthogonal to $A_i$.

Step 4. At time $\tau \in [t_i, t_{i+1}]$, write the firefront as $\{\gamma_i(\tau))\}$, where $\gamma_i(t)$ are obtained in step 3;

Step 5. Find the strategic path as the fire ray $\gamma_i(t)$ such that:

Case i. $||\gamma_i(\tau) - \gamma_i(0)||_{\text{Euc}}$ is the maximum among all the fire rays initiating from $A_i$, for the strategic path that maximizes the burnt area until time $\tau$;

Case ii. $\gamma_i(\tau) = q \in B$, in which $\tau$ is the first time when the fire reaches $B$, for the strategic path reaching a specific area $B$.

### 3.3 Some Discussion on Blocking Problem

It is essential to identify strategic paths to mitigate the impact of wildfires and prevent them from reaching certain areas. These paths are crucial for firefighting efforts but can also be dangerous due to the rapid spread of headfires and the limited effectiveness of small barriers. Therefore, larger barriers must be constructed in real time to block the fire's advance. These barriers, which can be created using various methods such as water drops from helicopters, vegetation clearance using bulldozers, or application of fire extinguishers by firefighter teams, effectively prevent the fire from crossing specific areas [3]. The objective is to minimize the total burned area and construction costs, a problem known as the blocking problem. Optimal strategies for constructing barriers have been studied, demonstrating the existence of optimal barriers [4]. This paper, by determining the strategic path, aids in identifying optimal locations for constructing barriers. We conjecture that barriers should be constructed orthogonally to the strategic paths, based on the associated Finsler metric, at the point where the strategic path intersects the barrier. However, this conjecture is not verified within the scope of this study. The size of barrier construction depends on operational strategies and available resources.

## 4 Example

We investigate hypothetical wildfire scenarios within *Ibitipoca State Park* in *Minas Gerais, Brazil*, where nearby residential areas pose a threat due to severe wildfires during dry winters. Simulating potential wildfires is crucial for effective management. We examine four wildfire propagations originating from $PC$[2] and want to see how the fire propagates and reaches the other nearby residential areas[3] with the wind expected to change twice in the next 24 hours. Two firefighter groups are dispatched: one to identify the first surrounded residential area and its corresponding path, and the other to identify the path of the fastest fire progression. We aim to find these paths to guide barrier construction to impede fire spread.

Using MATLAB, we implement our results, depicted in Figs. 1a-1d, showing firefronts and strategic paths. The black dashed paths represent routes where the fire surrounds residential areas

[2] *Pousada e Camping Canto da Vida*

[3] *Estância da Serra Ibitipoca*, *Quinta Do Barao Pousada*, *Pousada Tangara*, and *Pousada Canela De Ema* are shown in Figs. 1a-1d with black house icons labeled E, Q, T, and C, respectively.

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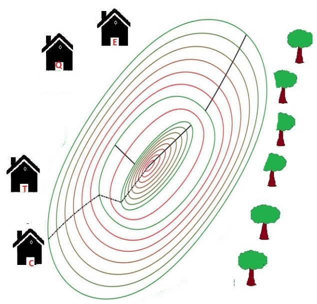
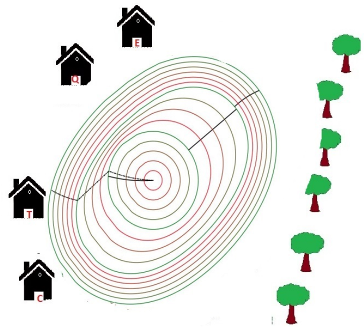
more rapidly, while thick black paths denote trajectories of faster fire progression during specific intervals. Each figure includes data sources.

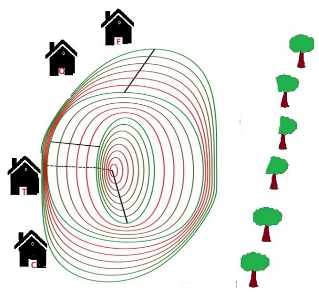
Upon observing Figs 1a-1d, it becomes evident that the first reached area by fire and the optimal strategic path for faster spread may vary. It highlights the importance of establishing a reliable propagation model. For instance, as illustrated in Fig. 1b, $T$ is the first area surrounded by fire, after 19 hours. If the firefighting team manages to construct a barrier within the initial 5 hours, it should be positioned along a direct path from $PC$ towards $C$. However, after 5 hours, this path does not work anymore, as the fire's propagation tendencies shift towards the southwest and then from toward northeast directions. In another case, Fig.1d, the fire engulfs both areas $Q$ and $T$, after 11 hours. The team must consider different paths for each interval to stop the fire toward $Q$.



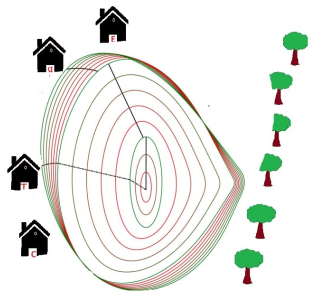
(a) $[(.5,0),1.1,3,\frac{\pi}{4},8]$ ; $[(0,.3),3,2,\frac{\pi}{6},5]$ ; $[.3(y,x),4,4,\frac{1}{3}(y-x/2),6]$.



(b) $[1.3(-y,x),4,4,.33(x-y/2),5]$; $[(.5,0),1.1,3,\frac{\pi}{4},8]$; $[.8(-y,x),2,2,\frac{y+x}{3},6]$.



(c) $[(.5,0),1.1,3,\frac{y}{4},8]$; $[(0,.3),3,2,\frac{x}{6},5]$; $[(.2,0),.5,y,\frac{\pi}{8},6]$.



(d) $[(0,.5),1.1,3,\frac{x}{3},5]$; $[(0,.2),3,2,y,3]$; $[.3(-y,-x),4,4,\frac{1}{3}(y+x),3]$.

Figure 1: Fire propagations near Ibitipoca State Park with time-space-varying winds and the data $[W,a,b,\theta,T]$ for each interval of time, Source: by the author.

## 5 Conclusion

We derived equations for firefronts and strategic paths, representing paths where the fire spreads faster or reaches specific locations. These equations apply to wildfire propagation in flat terrain under time-space-varying winds, enhancing propagation models' accuracy and improving fire behavior predictions. Strategic paths aid in determining barrier locations to reduce burnt areas or block fire progression, addressing what is known as the blocking problem in the literature.

To demonstrate the practical utility of our work, we implemented several scenarios of hypothetical wildfire propagations using MATLAB. Our experimental results illustrate the effectiveness of our approaches in positively impacting wildfire management strategies and resource allocation for fire control efforts.

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