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Lattice Index Coding from Construction π_A Lattices

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Abstract. This paper investigates Construction π_A lattices over $\mathbb{Z}$ applied to index coding which is used for efficiency in transmissions when the receiver may have side information. Key parameters as code size and side information gain are analised.

Index Terms. Lattices, Lattice codes, Chinese remainder theorem, Index coding.

1 Introduction

Lattices, viewed as algebraic structures, can be defined as discrete additive subgroups in the n-dimensional Euclidean space. They have been applied in different fields, particularly in communications, [1, 2]. In this context, it is useful to consider their construction from linear codes over the finite rings $\mathbb{Z}_q$. A technique to obtain lattices, introduced in [5], provided from an isomorphism guaranteed by the Chinese Remainder Theorem (CRT), is called Construction π_A. The multilevel nature of this construction brings the advantage of multistage decoding, reducing the complexity of the decoding process.

This paper aims to explore the application of Construction π_A over $\mathbb{Z}$ to the index coding problem and the associated side information gain. Index coding uses communication techniques that consider receiver side information to enhance transmission efficiency. Recent studies have explored the application of lattices in this framework, revealing improvements in decoding processes and good side information gain [3, 4, 6].

The work is organized as follows: In Section 2, we provide an overview of lattice codes and the main parameters for the index coding problem. Section 3 introduces the Construction π_A over $\mathbb{Z}$, as proposed in [5], aiming its applications to lattice index codes. Finally, Section 4 draws our conclusions and perspectives for future exploration.

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2 Preliminaries

In this section we summarize some concepts and properties related to lattices codes [1, 2] and we state the index coding problem [6].

2.1 Lattice codes

The next results and definitions can be found in references [1, 2].

A *lattice* is a subset of $\mathbb{R}^n$ generated by all integer combinations of linear independent vectors $v_1, \cdots, v_m$, m is called the lattice *rank* and we deal here only with full-rank lattices ($m = n$). A matrix B whose columns are the vectors v_i is called a *generator matrix* of Λ. The *volume* of a lattice, is given by $\mathrm{vol}(\Lambda) = |\det(B)|$.

Given a point $z \in \mathbb{R}^n$ and a lattice $\Lambda \subset \mathbb{R}^n$, we define $Q_\Lambda(z)$ as the *closest lattice point* to z,

$$Q_\Lambda(z) = x \in \Lambda;\ ||z - x|| \leq ||z - y||, \forall y \in \Lambda, \tag{1}$$

and ties must be chosen. *The Voronoi region* of a lattice Λ, $\mathcal{V}_\Lambda$, is the set of all points in $\mathbb{R}^n$ that are mapped to the origin under Q_Λ, $\mathcal{V}_\Lambda(0) = \mathcal{V}_\Lambda = \{z \in \mathbb{R}^n; \|z\| \leq \|z - y\|\ \forall y \in \Lambda\}$. Considering a sublattice $\Lambda' \subset \Lambda$, the set Λ/Λ' is called a *Voronoi constellation* and it is associated to the elements of Λ inside the Voronoi region of Λ'.

The *minimum distance* $d_{\min}(\Lambda)$ and the *center density* $\delta(\Lambda)$ of a lattice Λ are defined, respectively, as

$$d_{\min}(\Lambda) = \min_{0 \neq x \in \Lambda} ||x|| \quad , \quad \delta(\Lambda) = \frac{(d_{\min}(\Lambda)/2)^n}{\mathrm{vol}(\Lambda)}, \tag{2}$$

and we always have an $x \in \Lambda, x \neq 0$ with $d_{\min}(\Lambda) = \|x\|$.

A *linear code* over $\mathbb{Z}_q$, the ring of integers modulo q, is a subset $\mathscr{C} \subset \mathbb{Z}_q^n$ closed under addition. For $q = p$ (prime number) an linear code is a vector subspace of dimension $k \leq n$, called an $(n, k)-$linear code. A method for constructing lattices from linear codes is the *Construction A* [1, 2]. Considering the mapping $\rho : \mathbb{Z} \to \mathbb{Z}_q$, the natural reduction ring homomorphism and $\sigma : \mathbb{Z}_q \to \mathbb{Z}$, the standard inclusion map, extended to vectors component-wise. Given a linear code $\mathscr{C} \subset \mathbb{Z}_q^n$, the Construction A lattice associated with $\mathscr{C}$, denoted by $\Lambda_A(\mathscr{C})$, is defined as $\Lambda_A(\mathscr{C}) = \rho^{-1}(\mathscr{C}) = \sigma(\mathscr{C}) + q\mathbb{Z}^n$. It is shown that $\Lambda_A(\mathscr{C})$ is a full-rank lattice, $q\mathbb{Z}^n \subset \Lambda_A(\mathscr{C}) \subset \mathbb{Z}^n$, $\mathrm{vol}(\Lambda) = q^n/M$, where M is the size of the code $\mathscr{C}$ and $d_{\min}(\Lambda_A(\mathscr{C})) = \min\{d_{\min}(\mathscr{C}), q\}$, [2].

2.2 Index Coding Problem

The classic problem of index coding involves a transmitter with r independent messages $w_1, \ldots,$ w_r and a broadcast channel subject to additive white Gaussian noise (AWGN). Each receiver requests a subset of messages while having prior knowledge of a different subset of messages as side information. The objective for the transmitter is to send a coded packet with the lowest possible rate to fulfill the demands of all receivers.

Consider an AWGN transmission channel with a single transmitter and a finite number of receivers aiming to decode all messages from the source. Let $S \subset \{1, \ldots, r\}$ denote the set of indexes of the messages $w_S = (w_j; j \in S)$, representing the values known to the receivers as *side information*. We assume that the source has r independent messages $w_1, \ldots, w_r$, each belonging to respective alphabets $W_1, \ldots, W_r$. The transmitter jointly encodes the information $w_1, \ldots, w_r$ into a codeword $x \in \mathscr{C}$, where $\mathscr{C} \subset \mathbb{R}^n$ is a Voronoi constellation.

Definition 1 ([6]). *A lattice index code for r messages consists of r lattice constellations Λ_1/Λ', $\ldots, \Lambda_r/\Lambda'$, and the injective linear encoder map $\rho : \Lambda_1/\Lambda' \times \cdots \times \Lambda_r/\Lambda' \to \mathscr{C}$ given by*

$$\rho(w_1, \ldots, w_r) = (w_1 + \cdots + w_r) \mod \Lambda' \tag{3}$$

where $w_j \in \Lambda_j/\Lambda'$ and $\mathscr{C}$ is the set of all possible values of the transmit symbol $x = \rho(w_1, \ldots, w_r)$.

The rate of the j^{th} message is denoted as $R_j = 1/n \log_2 |W_j|$ bits per dimension, $j = 1, \ldots, r$.

Let $y = x+z$ be the received word, where $x \in \mathscr{C}$ and z is a random Gaussian vector representing noise. At the decoding stage, the receiver possesses knowledge of the messages $w_j = a_j \in S$. Thus, the receiver reduces the decoding search to the subcode $\mathscr{C}_{a_S} \subset \mathscr{C}$, excluding all codewords in $\mathscr{C}$ not corresponding to $w_S = a_S$. We define the minimum distance between any two codewords in $\mathscr{C}_{a_S}$ as $d_{a_S} = \min\{||x_i - x_j||; x_i, x_j \in \mathscr{C}_{a_S}, i \neq j\}$ and $d_S = \min\{d_{a_S}; a_j \in W_j, j \in S\}$ is the minimum of d_{a_S} over all possible values a_S of side information w_S. Therefore, when $S = \varnothing$, the minimum distance d_0 is the smallest distance between any pair of points in $\mathscr{C}$, $d_0 = d_{\min}(\mathscr{C})$.

Each bit per dimension of side information provides an information gain $\Gamma(\mathscr{C}, S)$ for the code $\mathscr{C}$ corresponding to the set of side information S, calculated as

$$\Gamma(\mathscr{C}, S) = \min_S \frac{10 \log_{10} \left(d_S^2/d_0^2\right)}{R_S} \text{ dB/b/dim.} \tag{4}$$

where $R_S = \sum_{j \in S} R_j$. This gain estimates the apparent signal-to-noise ratio (SNR) gain normalized by the side information transmission rate [6].

3 Construction π_A lattices

The Construction π_A lattice is a special case of Construction A lattice. Note that the proposed construction can be used to generate lattices over $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$, number fields, natural orders, and Hurwitz integers [3–5, 8]. It strongly depends on the existence of a ring isomorphism guaranteed by the well-known Chinese Remainder Theorem (CRT).

Proposition 1 ([5]). *Let $p_1, \ldots, p_k$ be a collection of distinct primes and let $q = \prod_{j=1}^{k} p_j$. There exists a ring isomorphism*

$$\varphi : \mathbb{Z}_q^n \to \mathbb{Z}_{p_1}^n \times \cdots \times \mathbb{Z}_{p_k}^n. \tag{5}$$

One way to obtain a ring isomorphism φ is to label every element $a \in \mathbb{Z}_q^n, q = \prod_{j=1}^{k} p_j$ by the natural mapping and then define $\varphi(a) = (a \mod p_1, \ldots, a \mod p_k)$.

To obtain the inverse of this, for $x_1, \ldots, x_k \in \mathbb{Z}$ and $m_j = q/p_j$, we can directly solve from the Bézout identity $\varphi^{-1}(c_1, \ldots, c_k) = x_1 m_1 c_1 + x_2 m_2 c_2 + \cdots + x_k m_k c_k \mod q\mathbb{Z}^n,\ (c_j \in \mathbb{Z}_{p_j}^n)$.

Definition 2 (Construction π_A [5]). *Let $p_1, \ldots, p_k$ be distinct primes and let $q = \prod_{j=1}^{k} p_j$. Consider l_j and n as integers such that $l_j \leq n$ and let G_j be a generator matrix of an (n, l_j)-linear code over $\mathbb{Z}_{p_j}$ for $j \in \{1, \ldots, k\}$. Construction π_A consists of the following steps,*

1. *Define the discrete codebooks $\mathscr{C}_j = \{x = G_j \cdot u : u \in \mathbb{Z}_{p_j}^{l_j}\}$ for $j \in \{1, \ldots, k\}$;*
2. *Construct $\mathscr{C} = \varphi^{-1}(\mathscr{C}_1 \times \cdots \times \mathscr{C}_k)$ where $\varphi^{-1} : \mathbb{Z}_{p_1}^n \times \cdots \times \mathbb{Z}_{p_k}^n \to \mathbb{Z}_q^n$ is a ring isomorphism;*
3. *$\mathscr{C}$ to the entire $\mathbb{R}^n$ to form $\Lambda_{\pi_A}(\mathscr{C}) = \sigma(\mathscr{C}) + q\mathbb{Z}^n$.*

3.1 Lattice index codes from Construction π_A over $\mathbb{Z}$

Considering linear codes $\mathscr{C}_1, \ldots, \mathscr{C}_r$ of ranks $k_1, \ldots, k_r$ over $\mathbb{Z}_{p_1}^n, \ldots, \mathbb{Z}_{p_r}^n$ respectively, the cardinality of each code is given by $|\mathscr{C}_j| = p_j^{k_j},\ j = 1, \ldots, r$. Thus, for $\mathscr{C} = \varphi^{-1}(\mathscr{C}_1 \times \cdots \times \mathscr{C}_r)$, the cardinality becomes $|\mathscr{C}| = |\mathscr{C}_1| \cdot \cdots \cdot |\mathscr{C}_r| = p_1^{k_1} \cdot \cdots \cdot p_r^{k_r}$. Moreover, via Construction A, we can obtain a lattice in each layer $\Lambda_A(\mathscr{C}_j) = \sigma(\mathscr{C}_j) + p_j\mathbb{Z}^n$. It's noteworthy that $\Lambda_j/p_j\mathbb{Z}^n \cong m_j\Lambda_j/q\mathbb{Z}^n$. Denoting $\Lambda_j := m_j\Lambda_A(\mathscr{C}_j)$ and $\Lambda' := q\mathbb{Z}^n$, the map φ restricted to the quotients can be written as,

$$\rho : \Lambda_1/\Lambda' \times \cdots \times \Lambda_r/\Lambda' \to \Lambda/\Lambda' \tag{6}$$

$$(v_1, v_2, \ldots, v_r) \mapsto (x_1 v_1 + \cdots + x_r v_r) \mod q\mathbb{Z}^n, \tag{7}$$

with $\mathscr{C} = \Lambda/\Lambda'$ being the set of all possible codewords $v = \rho(v_1, \ldots, v_r)$. Since φ is an isomorphism, it ensures a unique decoding for each codeword.

Now, based on the same analysis of [2, 6], let us relate the lattice index coding properties with its components in this case.

The rate of the j^{th} message is given by, $R_j = \frac{1}{n} \log_2 |\Lambda_j/\Lambda'| = \frac{1}{n} \log_2 \left(\frac{\mathrm{vol}(\Lambda')}{\mathrm{vol}(\Lambda_j)}\right)$ bit/dim.

The subcodes can be characterised as,

$$\mathscr{C}_{a_S} = \{\rho(w_1, \ldots, w_r); w_j = a_j, j \in S \text{ and } w_j \in \mathscr{C}, j \notin S\} \subset \mathscr{C} \tag{8}$$

in terms of the lattices Λ_j, assuming the receiver has knowledge of the side information $w_j = a_j$, for $j \in S$. Denoting the complement of S as S^c and let Λ_{S^c} be the lattice obtained by the sum of

$\Lambda_j, j \in S^c$, then, the subcode used for decoding at the receiver is,

$$\mathscr{C}_{a_S} = \left\{ \left(\sum_{j \in S} x_j a_j + \Lambda_{S^c} \right) \mod q\mathbb{Z}^n \right\}. \tag{9}$$

that is, $\mathscr{C}_{a_S}$ is obtained by the translation of the Voronoi constellations Λ_{S^c}/Λ' by the vector $\sum_{j \in S} x_j a_j \mod q\mathbb{Z}^n$, and the minimum distance in $\mathscr{C}_{a_S}$ is $d_S = d_{\min}(\Lambda_{S^c})$.

The next proposition can be obtained using arguments similar to the ones in ([6], *Lemma 2*).

Proposition 2. *For the lattices $\Lambda_1, \ldots, \Lambda_r$ and Λ' as defined in (7), the following is valid:*

(i) The map ρ generates a lattice index coding with $\mathscr{C} = \Lambda/\Lambda'$;

(ii) $vol(\Lambda_{S^c}) = \prod_{j \in S} p_j^{k_j} vol(\Lambda)$;

(iii) $d_0 \leq d_S \leq \prod_{j \in S} p_j d_0$.

3.2 The side information gain

To analyse the side information gain, let us assume S as the set of side information and $d_S = d_{\min}(\Lambda_{S^c})$. Since $R = R_1 + \cdots + R_r = \frac{1}{n} \sum_{j=1}^{r} \log_2 |\Lambda_j/\Lambda'|$, we have, $nR = \log_2 \left(\prod_{i=1}^{r} |\Lambda_i/\Lambda'| \right)$ $\Rightarrow 2^{nR} = |\Lambda_1/\Lambda'| \cdot \cdots \cdot |\Lambda_r/\Lambda'|$.

Since the messages are uniquely mapped, $R = \frac{1}{n} \log_2 |\Lambda/\Lambda'|$. Thus, the rate for this receiver is,

$$R_S = R - \sum_{j \in S^c} R_j = \frac{1}{n} \log_2 \left(\frac{\mathrm{vol}(\Lambda')}{\mathrm{vol}(\Lambda)} \right) - \frac{1}{n} \log_2 \left(\frac{\mathrm{vol}(\Lambda')}{\mathrm{vol}(\Lambda_{S^c})} \right) = \frac{1}{n} \log_2 \left(\frac{\mathrm{vol}(\Lambda_{S^c})}{\mathrm{vol}(\Lambda)} \right) \tag{10}$$

$\dfrac{\mathrm{vol}(\Lambda_{S^c})}{\mathrm{vol}(\Lambda)} = \dfrac{q^n / \prod_{j \in S^c} p_j^{k_j}}{q^n / \prod_{i=1}^{r} p_i^{k_i}} = \prod_{j \in S} p_j^{k_j}$, and then $R_S = \log_2 \left(\prod_{j \in S} p_j^{k_j/n} \right)$. The side information gain is given by

$$\Gamma(\mathscr{C}, S) = \min_S \frac{10 \log_{10}(d_S^2/d_0^2)}{R_S} = \min_S 20 \log_{10} 2 \cdot \frac{\log_{10}(d_S/d_0)}{\log_{10} \left(\prod_{j \in S} p_j^{k_j/n} \right)} \tag{11}$$

We can also write the volume ratio in (10) in terms of center density,

$$R_S = \frac{1}{n} \log_2 \left(\frac{d_{\min}(\Lambda_{S^c})^n}{\delta(\Lambda_{S^c})} \cdot \frac{\delta(\Lambda)}{d_{\min}(\Lambda)^n} \right) = \log_2 \left(\frac{d_S}{d_0} \right) + \frac{1}{n} \log_2 \left(\frac{\delta(\Lambda)}{\delta(\Lambda_{S^c})} \right). \tag{12}$$

If $\delta(\Lambda) \geq \delta(\Lambda_{S^c})$, then $R_S \geq \log_2 (d_S/d_0)$ and an upper bound can be derived [6],

$$\Gamma(\mathscr{C}, S) = \min_S \frac{10 \log_{10}(d_S^2/d_0^2)}{R_S} \leq \frac{20 \log_{10}(d_S/d_0)}{\log_2(d_S/d_0)} = 20 \log_{10} 2 \approx 6\mathrm{dB/bit/dim}. \tag{13}$$

In our approach, we do not have always that $\delta(\Lambda) \geq \delta(\Lambda_{S^c})$, but its also possible to derive an upper bound for $\Gamma(\mathscr{C}, S)$. From *Proposition* 2, (11) and since $\log_{10}(d_S/d_0) \leq \log_{10} \left(\prod_{j \in S} p_j \right) \leq \log_{10} \left(\prod_{j \in S} p_j^{k_j} \right)$ we have,

$$\Gamma(\mathscr{C}, S) \leq 20\log_{10} 2 \cdot \frac{\log_{10}(d_S/d_0)}{\log_{10}\left(\prod_{j\in S} p_j^{k_j/n}\right)} \leq 20\log_{10} 2 \cdot \frac{\log_{10}\left(\prod_{j\in S} p_j^{k_j}\right)}{\log_{10}\left(\prod_{j\in S} p_j^{k_j/n}\right)} = n \cdot 20\log_{10} 2, \tag{14}$$

and this is an upper bound for side information gain to lattice index coding from Construction π_A over $\mathbb{Z}$. If for all S, $\delta(\Lambda) < \delta(\Lambda_{S^c})$ then $6 \leq \Gamma(\mathscr{C}, S) \leq n \cdot 20\log_{10} 2$ dB/bit/dim, but is higher side information gain will be attached to a lower packing rate for Λ compared to Λ_{S^c} and therefore some inefficiency for the AWGN channel, [6].

Example 1. *Consider the following isomorphism for Construction π_A lattices,*

$$\varphi^{-1} : \mathbb{Z}_2^2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_5^2 \to \mathbb{Z}_{30}^2 \tag{15}$$

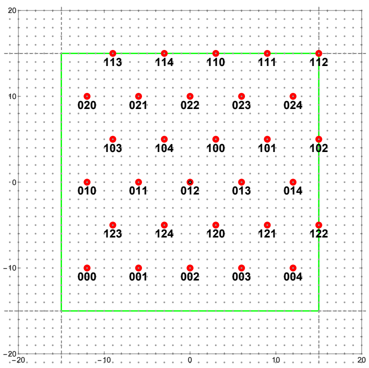
$$(w_1, w_2, w_3) \mapsto (15w_1 + 10w_2 + 6w_3) \mod 30\mathbb{Z}^2 \tag{16}$$

Let $G_1 = (1\ 1)^T, G_2 = (0\ 2)^T$ and $G_3 = (3\ 0)^T$ be the generator matrices of the codes $\mathscr{C}_1 \subset \mathbb{Z}_2^2, \mathscr{C}_2 \subset \mathbb{Z}_3^2$ and $\mathscr{C}_3 \subset \mathbb{Z}_5^2$, respectively. Then, by Construction π_A we can obtain the lattice $\Lambda = \Lambda_{\pi_A}(\mathscr{C})$, where $\mathscr{C} = \varphi^{-1}(\mathscr{C}_1 \times \mathscr{C}_2 \times \mathscr{C}_3) = \langle (3\ \ 5)^T \rangle \subset \mathbb{Z}_{30}^2$. Now, taking $\Lambda_1 = 15\Lambda_A(\mathscr{C}_1), \Lambda_2 = 10\Lambda_A(\mathscr{C}_2), \Lambda_3 = 6\Lambda_A(\mathscr{C}_3)$ and $\Lambda' = 30\mathbb{Z}^2$ we can restrict the mapping φ^{-1} to obtain the lattice index code,

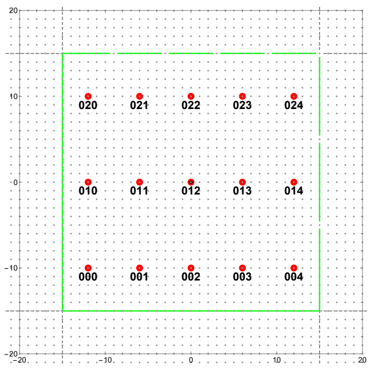
$$\varphi^{-1} : \Lambda_1/\Lambda' \times \Lambda_2/\Lambda' \times \Lambda_3/\Lambda' \to \Lambda/\Lambda' \tag{17}$$

$$(v_1, v_2, v_3) \mapsto \mathscr{C} = (v_1 + v_2 + v_3) \mod 30\mathbb{Z}^2 \tag{18}$$

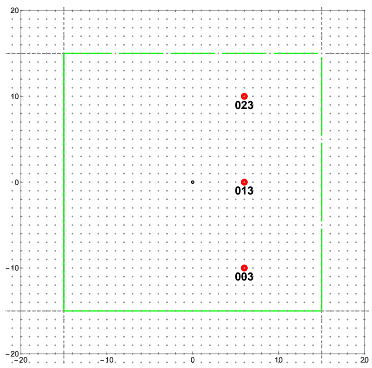
where $v_j \in \Lambda_j/\Lambda'$ and $\mathscr{C} = \Lambda/\Lambda'$. In Figure 1 it is illustrated the code $\mathscr{C}$ and examples of subcodes when the receiver has the knowledge of the values of some prior information.



(a)



(b)



(c)

Figure 1: In Figure (a) we have the code $\mathscr{C} = \Lambda/\Lambda'$; (b) and (c) presents the subcodes $\mathscr{C}_{a_1} \subset \mathscr{C}$ where $S = \{1\}$ and $w_1 = 0$, $\mathscr{C}_{a_{1,3}} \subset \mathscr{C}$ where $S = \{1, 3\}$, $w_1 = 0$ and $w_3 = 4$, respectively.

Note that in this case, $\delta(\Lambda) = \frac{(\sqrt{34}/2)^2}{30} \approx 0.28333$ and for all set of index S we have $\delta(\Lambda) > \delta(\Lambda_{S^c})$ then $\Gamma(\mathscr{C}, S) \leq 6$ dB/bit/dim.

4 Conclusions and perspectives

We have shown that the use of Construction π_A in lattice index coding naturally emerges from the construction proposed in [6]. This application expands the possibility for designing lattice index codes. Additionally, we can explore the decoding benefits derived from the multilevel nature of Construction π_A, as in [5, 8]. Promising avenues for further research include extending the analysis to Construction π_A lattices over the Gaussian and Eisenstein integers [7], more general ring of integers over number fields and to the maximal order of Hurwitz integers, exploring the decoding advantages when the receiver has prior knowledge.

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