Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 11, n. 1, 2025.

Trabalho apresentado no XLIII CNMAC, Centro de Convenções do Armação Resort - Porto de Galinhas - PE, 2024

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Constructing Fuzzy Material Implication Functions Derived from General Grouping Functions

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Abstract. Grouping and overlap functions have been largely applied in the modeling of fuzzy systems and problems involving decision-making based on fuzzy preference relations due to their richness in the classes of aggregation functions compared to t-conorms and t-norms. Grouping functions allow one to measure the amount of evidence favoring two given alternatives in pairwise comparisons. However, as they are not associative, in the context of n-dimensional problems, some generalizations of grouping functions are required, like n-dimensional grouping functions and the more flexible class called general grouping functions (GGF). Since GGF widens the scope of applications, a novel class of fuzzy implication functions constructed from GGF and fuzzy negations is provided in this work. We study their main properties, characterizations, construction methods, and examples, paving the way for their use in modeling more flexible fuzzy systems.

Keywords. General Grouping Functions, Fuzzy Implication Functions, System Modeling

1 Introduction

Compared to well-known aggregation functions such as t-norms and t-conorms [13], overlap and grouping functions are richer aggregation functions. They present the self-closeness property concerning the convex sum and the aggregation by the composition of overlap and/or grouping functions [10], whereas t-norms and t-conorms do not. Also, the maximum t-conorm is the only idempotent t-conorm and the unique homogeneous t-conorm. Nevertheless, there are many idempotent and homogeneous grouping functions. However, since overlap and grouping functions may be non-associative, their extension to n-ary functions is not so immediate. They were just applied in bi-dimensional problems (when only a pair of classes are considered), up to the study seen in [11], where n-dimensional overlap functions were introduced and applied to fuzzy rule-based classification systems (FRBCSs). In [12], n-dimensional grouping functions are defined and applied to quantify the quality of a fuzzy community detection output based on n-dimensional operators.

A more flexible concept in the n-dimensional context was given in [9], where general overlap functions (GOF) were defined with less restrictive boundary conditions and applied to identify the matching degree in the fuzzy reasoning method of FRBCSs. Analogously, in [17], we have the theoretical basis of general grouping functions (GGF), a resource that allows more flexibility

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to n-dimensional grouping functions. They can be interpreted, for example, as the quantity of evidence in favor of one alternative among multiple ones when performing n-ary comparisons in multi-criteria decision-making based on n-ary fuzzy heterogeneous, incomplete preference relations.

GOF and GGF definitions pave the way to research properties, related concepts, and extensions to the interval-valued context [3]. Therefore, one question arises: What would be the role of fuzzy material implication functions derived from GGF and fuzzy negations?

If-then rules in fuzzy rule-based systems make the process of representing inferential knowledge very intuitive and, therefore, is a commonly used strategy in several works. It is possible to construct implication-like operators in distinct ways. In the fuzzy logic setting, implication functions have been deeply investigated in the applied and theoretical fields [5]. Some works on fuzzy implication functions (FIF, for short) offer less restrictive operators than t-norms and t-conorms, e.g., the works using (i) uninorms under some conditions [18]; (ii) copulae [1]; (iii) pseudo-triangular norms [14]; (iv) semi-copulae [4]; and, (v) aggregation functions [15]. In particular, in this paper, we study material implication functions derived from GGF and fuzzy negations [6]. We use GGF to generalize the Boolean implication $p \to q \equiv \sim p \vee q$, studying their properties and characterizations.

The main objective of the present work is to introduce a new and more flexible class of fuzzy material implications, namely the (GG, N)-implication functions derived from general grouping functions GG and fuzzy negations N, studying properties and providing their characterization (Sect. 3). Moreover, Sect. 2 presents some preliminary concepts, and Sect. 4 has the Conclusion.

2 Preliminaries

In this section, we highlight some relevant concepts used in this work.

Definition 2.1. *[7] A function $N\colon [0,1] \to [0,1]$ is called **fuzzy negation** if the following two conditions hold, $\forall x, y \in [0,1]$: **(N1)** $N(x) \leq N(y)$ if $y \leq x$, i.e., N is decreasing, and **(N2)** $N(0) = 1$ and $N(1) = 0$ (boundary conditions). It is **strict** if **(N3)** N is continuous and $N(x) > N(y)$ whenever $y > x$. It is strong if **(N4)** $N(N(x)) = x$.*

Definition 2.2. *[5] A binary operator $I\colon [0,1]^2 \to [0,1]$ is said to be a FIF, if the following conditions hold, $\forall x, y, z \in [0,1]$: **(I1)** If $x \leq z$ then $I(x, y) \geq I(z, y)$; **(I2)** If $y \leq z$ then $I(x, y) \leq I(x, z)$; **(I3)** $I(0, y) = 1$; **(I4)** $I(x, 1) = 1$; **(I5)** $I(1, 0) = 0$.*

Remark 2.1. *In order to obtain an equivalent definition regarding Def. 2.2, one can substitute conditions **(I3)** and **(I4)**, respectively, by: **(I3*)** $I(0, 0) = 1$ and **(I4*)** $I(1, 1) = 1$.*

Definition 2.3. *[5] Take $I : [0,1]^2 \to [0,1]$ as a FIF. The function $N_I\colon [0,1] \to [0,1]$, given, $\forall x \in [0,1]$, by $N_I(x) = I(x, 0)$, is called the natural negation of I or the negation induced by I.*

Definition 2.4. *[5] Some properties may be studied for a FIF $I\colon [0,1]^2 \to [0,1]$, $\forall x, y, z \in [0,1]$:*

(NP) ***Left neutrality property:*** *$I(1, y) = y$;*

(EP) ***Exchange principle:*** *$I(x, I(y, z)) = I(y, I(x, z))$;*

(LI) ***Law of importation*** *with a t-norm T: $I(T(x, y), z) = I(x, I(y, z))$;*

(CP) ***Law of contraposition*** *with respect to a fuzzy negation N: $I(x, y) = I(N(y), N(x))$;*

(L-CP) ***Left contraposition law*** *with respect to a fuzzy negation N: $I(N(x), y) = I(N(y), x)$;*

(R-CP) ***Right contraposition law*** *with respect to a fuzzy negation N: $I(x, N(y)) = I(y, N(x))$.*

Definition 2.5. *[7] A function $A\colon [0,1]^n \to [0,1]$ is said to be an aggregation function (AF), if, $\forall x_1, \ldots, x_n \in [0,1]$: **(A1)** A satisfies: $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$; **(A2)** A is increasing, i.e., for each $i \in \{1, \ldots, n\}$, if $x_i \leq y$ then $A(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$.*

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Definition 2.6. *[13] Let $S\colon [0,1]^2 \to [0,1]$ be a binary AF, then S is a triangular conorm (t-conorm), if, $\forall x, y, z \in [0,1]$: **(S1)** Commutativity: $S(x,y) = S(y,x)$; **(S2)** Associativity: $S(x, S(y,z)) = S(S(x,y),z)$; **(S3)** Increasingness: if $x \leq y$ then $S(x,z) \leq S(y,z)$; **(S4)** Neutral element: $S(x,0) = x$.*

Definition 2.7. *[11] A mapping $G_n\colon [0,1]^n \to [0,1]$ is an n-dimensional grouping function if, $\forall \vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$: ($G_{n1}$) G_n is commutative; (G_{n2}) $G_n(\vec{x}) = 0$ iff $x_i = 0$, $\forall i = 1, \ldots, n$; (G_{n3}) $G_n(\vec{x}) = 1$ iff $\exists i \in \{1, \ldots, n\}$ with $x_i = 1$; (G_{n4}) G_n is increasing; (G_{n5}) G_n is continuous.*

Henceforth, we will refer to binary grouping functions directly as grouping functions [8].

Definition 2.8. *[17] A function $GG\colon [0,1]^n \to [0,1]$ is said to be a GGF if, $\forall \vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$: **(GG1)** GG is commutative; **(GG2)** If $\sum_{i=1}^{n} x_i = 0$, then $GG(\vec{x}) = 0$; **(GG3)** If $\exists i \in \{1, \ldots, n\}$ such that $x_i = 1$, then $GG(\vec{x}) = 1$; **(GG4)** GG is increasing; **(GG5)** GG is continuous.*

Definition 2.9. *[2] A function $G_n\colon [0,1]^n \to [0,1]$ is said to be n-dimensional 0-grouping function if the condition (G_{n2}) in Def. 2.7 is replaced by: ($G_{n2'}$) If $x_i = 0$, $\forall i = 1, \ldots, n$, then $G_n(\vec{x}) = 0$. Likewise, a function $G_n\colon [0,1]^n \to [0,1]$ is said to be an n-dimensional 1-grouping function if the condition (G_{n3}) is replaced by: ($G_{n3'}$) If there exists $i \in \{1, \ldots, n\}$ with $x_i = 1$, then $G_n(\vec{x}) = 1$.*

Remark 2.2. *Observe that non-0-positive bivariate grouping and non-1-positive bivariate grouping functions were originally called as 0-grouping and 1-grouping functions [16], respectively. Also, if $G_n\colon [0,1]^n \to [0,1]$ is an n-dimensional grouping function, 0-grouping or 1-grouping function, then G_n is also a GGF.*

3 Material Fuzzy Implications Derived from GGF

Dimuro et al. [10] introduced (G,N)-implications, defined from the composition of a grouping function and a fuzzy negation. Inspired by that, our study replaces the grouping function with a bivariate GGF to introduce a new class of FIF called (GG,N)-implication function.

Definition 3.1. *Take a GGF $GG\colon [0,1]^2 \to [0,1]$ and let N be a fuzzy negation. The mapping $I_{GG,N}\colon [0,1]^2 \to [0,1]$ is defined, $\forall x, y \in [0,1]$, by $I_{GG,N}(x,y) = GG(N(x), y)$.*

Proposition 3.1. *The function $I_{GG,N}\colon [0,1]^2 \to [0,1]$ is a FIF, entitled (GG,N)-implication.*

Proof. For a GGF $GG\colon [0,1]^2 \to [0,1]$ and a fuzzy negation N, let us verify if, $\forall x, y, z \in [0,1]$, the function $I_{GG,N}\colon [0,1]^2 \to [0,1]$ defined according to Def. 3.1 satisfies the conditions from Def. 2.2:

- **(I1)** Bearing in mind that condition **(GG4)** holds for GG, then $x \leq y \overset{\textbf{(N1)}}{\Rightarrow} N(x) \geq N(y) \Rightarrow GG(N(x), z) \geq GG(N(y), z)$. Therefore, $I_{GG,N}(x,z) \geq I_{GG,N}(y,z)$;
- **(I2)** As **(GG4)** holds for GG, we have $y \leq z \Rightarrow GG(N(x), y) \leq GG(N(x), z)$. So, $I_{GG,N}(x,y) \leq I_{GG,N}(x,z)$;
- **(I3)** By **(GG3)**, one has that $I_{GG,N}(0,y) = GG(N(0), y) \overset{\textbf{(N2)}}{=} GG(1,y) = 1$;
- **(I4)** By **(GG3)**, we have that $I_{GG,N}(x,1) = GG(N(x), 1) = 1$;
- **(I5)** By **(GG2)**, one has that $I_{GG,N}(1,0) = GG(N(1), 0) \overset{\textbf{(N2)}}{=} GG(0,0) = 0$.

Therefore, $I_{GG,N}$ is a FIF. □

The following result ensures that the class of (GG,N)-implication functions, where GG does not have 0 as a neutral element, does not intersect with the class of (S,N)-implications.

Proposition 3.2. *If a GGF GG does not have 0 as neutral element, then $I_{GG,N} \neq I_{S,\widetilde{N}}$ for any t-conorm S and any fuzzy negations N and $\widetilde{N}$ (N and $\widetilde{N}$ not being necessarily the same).*

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Proof. Given a GGF GG and a fuzzy negation N, suppose that there exists a t-conorm S and a fuzzy negation $\widetilde{N}$, such that $I_{GG,N}(x,y) = I_{S,\widetilde{N}}(x,y)$, $\forall x, y \in [0,1]$. For $x = 1$, $GG(N(1), y) = S(\widetilde{N}(1), y)$. Therefore, $GG(0,y) = S(0,y) = y, \forall y \in [0,1]$, contradiction. □

Proposition 3.3. *Let GG and N be a GGF and a fuzzy negation, respectively. Then,*

(i) If 0 is the neutral element of GG, then $N_{I_{GG,N}} = N$;

(ii) If N is strict and $N_{I_{GG,N}} = N$, then 0 is the neutral element of GG.

Proof. (i) Once 0 is the neutral element of GG, then, $\forall x \in [0,1]$, it holds that $N_{I_{GG,N}}(x) = I_{GG,N}(x,0) = GG(N(x),0) = N(x)$. (ii) Since N is strict, then we have that, $\forall x \in [0,1]$, it holds that $GG(x,0) = GG(N(N^{-1}(x)),0) = I_{GG,N}(N^{-1}(x),0) = N_{I_{GG,N}}(N^{-1}(x))$. By hypothesis, take $N_{I_{GG,N}} = N$. Thus, $GG(x,0) = N(N^{-1}(x)) = x$, $\forall x \in [0,1]$. So, 0 is the GG neutral element. □

Note that there are non-strict fuzzy negations N that satisfy $N_{I_{GG,N}} = N$, but GG has no neutral element, i.e., the converse of Prop. 3.3(i) is not always true.

Proposition 3.4. *Let $GG\colon [0,1]^2 \to [0,1]$ be a GGF, $N_1, N_2\colon [0,1] \to [0,1]$ be fuzzy negations and $I_{GG,N_1}, I_{GG,N_2}\colon [0,1]^2 \to [0,1]$ be (GG,N)-implication functions. Then it holds that:*

(i) If $N_1 \leq N_2$ then $I_{GG,N_1} \leq I_{GG,N_2}$;

(ii) If GG has a neutral element n and $I_{GG,N_1} \leq I_{GG,N_2}$, then $N_1 \leq N_2$.

Proof. (i) Since $N_1 \leq N_2$, by **(GG4)** we have that $GG(N_1(x),y) \leq GG(N_2(x),y)$ $\forall x,y \in [0,1]$, i.e., $I_{GG,N_1} \leq I_{GG,N_2}$. (ii) Since $I_{GG,N_1} \leq I_{GG,N_2}$, so, in particular, $GG(N_1(x),n) \leq GG(N_2(x),n)$, $\forall x \in [0,1]$. Therefore, $N_1(x) \leq N_2(x)$ $\forall x \in [0,1]$, once n is a neutral element of GG. □

Proposition 3.5. *Let $GG_1, GG_2\colon [0,1]^2 \to [0,1]$ be GGF, $N_1, N_2\colon [0,1] \to [0,1]$ be fuzzy negations and $I_{GG_1,N_1}, I_{GG_2,N_2}\colon [0,1]^2 \to [0,1]$ be (GG,N)-implication functions. If $I_{GG_1,N_1} \leq I_{GG_2,N_2}$ then:*

(i) If $a_1 \leq a_2$, where a_i is a neutral element of GG_i, for $i \in \{1,2\}$, then $N_1 \leq N_2$;

(ii) If $N_1 = N_2$ is continuous, then $GG_1 \leq GG_2$.

Proof. (i) Considering $I_{GG_1,N_1} \leq I_{GG_2,N_2}$ and $a_1 \leq a_2$, thus $GG_1(N_1(x),a_1) \leq GG_2(N_2(x),a_2)$, $\forall x \in [0,1]$. So, since a_i is a neutral element of GG_i, for $i \in \{1,2\}$, $N_1(x) \leq N_2(x)$, $\forall x \in [0,1]$. (ii) Since $N_1 = N_2 = N$ is continuous, for each $x \in [0,1]$ there is $\widetilde{x} \in [0,1]$ such that $N(\widetilde{x}) = x$. So, $GG_1(x,y) = GG_1(N(\widetilde{x}),y) \leq GG_2(N(\widetilde{x}),y) = GG_2(x,y)$, $\forall x,y \in [0,1]$. Thus, $GG_1 \leq GG_2$. □

Proposition 3.6. *Let $GG\colon [0,1]^2 \to [0,1]$ be a GGF and N be a fuzzy negation. Then, $y \leq GG(0,y)$ iff $y \leq I_{GG,N}(x,y)$, $\forall x,y \in [0,1]$, where $I_{GG,N}\colon [0,1]^2 \to [0,1]$ is a (GG,N)-implication.*

Proof. If $y \leq GG(0,y)$, then, since $0 \leq N(x)$, $\forall x \in [0,1]$, and **(GG4)**, we have that $y \leq GG(0,y) \leq GG(N(x),y) = I_{GG,N}(x,y)$. Conversely, if $y \leq I_{GG,N}(x,y)$, $\forall x \in [0,1]$, then, in particular, for $x = 1$, $y \leq GG(N(1),y) = GG(0,y)$. □

Corollary 3.1. *Let $GG\colon [0,1]^2 \to [0,1]$ be a GGF. Take N as a fuzzy negation and $I_{GG,N}\colon [0,1]^2 \to [0,1]$ as a (GG,N)-implication. If 0 is GG neutral element, then $y \leq I_{GG,N}(x,y)$, $\forall x,y \in [0,1]$.*

Lemma 3.1. *Let GG be a GGF. If GG is associative, then 0 is the neutral element of GG.*

Proof. Since GG is associative, $\forall x,y,z \in [0,1]$, one has that $GG(x, GG(y,z)) = GG(GG(x,y),z)$. In particular, for $x = y = 0$, and by **(GG2)**, $GG(0, GG(0,z)) = GG(0,z)$, $\forall z \in [0,1]$. Now, given any $y \in [0,1]$, as GG is continuous, there exists $z \in [0,1]$, such that $GG(0,z) = y$. Thus, one has that $GG(0,y) = GG(0, GG(0,z)) = GG(0,z) = y$. Therefore, 0 is the neutral element of GG. □

Proposition 3.7. *Let GG be a GGF. If GG is associative, then GG is a t-conorm.*

Proof. It is immediate from **(GG1)** and **(GG4)**, GG associativity and Lemma 3.1. □

Corollary 3.2. *Let GG be a GGF. If GG is associative, then $I_{GG,N}$ is an (S,N)-implication.*

The next propositions study under which conditions, (GG,N)-implications satisfy some of the properties of implication functions provided in Def. 2.4.

Proposition 3.8. *Let $I_{GG,N}\colon [0,1]^2 \to [0,1]$ be a (GG,N)-implication function, so:*

(i) $I_{GG,N}$ satisfies (NP) if and only if 0 is the neutral element of GG.

(ii) If GG is associative, then $I_{GG,N}$ satisfies (EP). Furthermore, if N is strict and $I_{GG,N}$ satisfies (EP), then GG is associative.

Proof. (i) $\forall y \in [0,1]$, one has that $I_{GG,N}(1,y) = y \Leftrightarrow y = GG(N(1),y) \overset{\textbf{(N2)}}{=} GG(0,y)$. (ii) By Proposition 3.7, GG is a t-conorm. So, the result follows straight from Proposition 2.4.3(i) in [5]. Now, consider that N is strict and $I_{GG,N}$ satisfies (EP). So, $\forall x,y,z \in [0,1]$, we have that:

$$\begin{aligned}GG(x,GG(y,z)) &\overset{\textbf{(GG1)}}{=} GG(x,GG(z,y)) = GG(N(N^{-1}(x)),GG(N(N^{-1}(z)),y))\\ &= I_{GG,N}(N^{-1}(x),I_{GG,N}(N^{-1}(z),y)) \overset{\textbf{(EP)}}{=} I_{GG,N}(N^{-1}(z),I_{GG,N}(N^{-1}(x),y))\\ &= GG(N(N^{-1}(z)),GG(N(N^{-1}(x)),y)) = GG(z,GG(x,y)) \overset{\textbf{(GG1)}}{=} GG(GG(x,y),z).\end{aligned} \tag{1}$$

Therefore, GG is associative. □

Proposition 3.9. *Let $I_{GG,N}\colon [0,1]^2 \to [0,1]$ be a (GG,N)-implication function. Then:*

(i) $I_{GG,N}$ satisfies the R-CP(N) property.

(ii) If N is strict, then $I_{GG,N}$ satisfies L-CP(N^{-1}).

(iii) If $I_{GG,N}$ satisfies L-CP(N) and 0 is the neutral element of GG, then N is strong.

(iv) If N is strong, then $I_{GG,N}$ satisfies CP(N).

(v) If $I_{GG,N}$ satisfies CP(N) and 0 is the neutral element of GG, then N is strong.

Proof. (i) Assuming **(GG1)** holds, $\forall x,y \in [0,1]$, $I_{GG,N}(x,N(y)) = GG(N(x),N(y)) = GG(N(y),N(x)) = I_{GG,N}(y,N(x))$. So, $I_{GG,N}$ holds R-CP(N). (ii) Since N is strict, by **(GG1)**, $I_{GG,N}(N^{-1}(x),y) = GG(N(N^{-1}(x)),y) = GG(x,y) \overset{\textbf{(GG1)}}{=} GG(y,x) = GG(N(N^{-1}(y)),x) = I_{GG,N}(N^{-1}(y),x)$, $\forall x,y \in [0,1]$. Then, $I_{GG,N}$ satisfies L-CP(N^{-1}). (iii) Since $I_{GG,N}$ holds L-CP(N), $GG(N(N(x)),y) = GG(N(N(y)),x)$, $\forall x,y \in [0,1]$. Particularly, for $y=0$, $GG(N(N(x)),0) = GG(N(N(0)),\ x) \overset{\textbf{(N2)}}{=} GG(0,x)$. So, since 0 is the neutral element of GG, one concludes $N(N(x)) = x$, $\forall x \in [0,1]$. Hence, N is strong. (iv) Since N is strong, $I_{GG,N}(N(y),N(x)) = GG(N(N(y)),N(x)) = GG(y,\ N(x))$, $\forall x,y \in [0,1]$. So, by **(GG1)**, we have that $I_{GG,N}(N(y),N(x)) = GG(N(x),y) = I_{GG,N}(x,y)$. So, $I_{GG,N}$ holds CP(N). (v) Since $I_{GG,N}$ satisfies CP(N), $GG(N(N(y)),N(x)) = GG(N(x),y)$, $\forall x,y \in [0,1]$. And, for $x{=}1$, by **(N2)**, $GG(N(N(y)),0) = GG(0,y)$. So, since 0 is the neutral element of GG, then $N(N(y)) = y$, $\forall y \in [0,1]$. Therefore, N is strong. □

Lemma 3.2. *Let $I\colon [0,1]^2 \to [0,1]$ be a continuous FIF. When $N_I\colon [0,1] \to [0,1]$ is a strict fuzzy negation and I satisfies L-CP(N_I^{-1}), then $GG_I(x,y) = I(N_I^{-1}(x),y)$ is a GGF.*

Proof. GG_I must satisfy all conditions from Def. 2.8, $\forall x,y,z \in [0,1]$. **(GG1)** As I satisfies L-CP(N_I^{-1}), $GG_I(x,y) = I(N_I^{-1}(x),y) = I(N_I^{-1}(y),x) = GG_I(y,x)$; **(GG2)** If $x=y=0$, then $GG_I(0,0) = I(N_I^{-1}(0),0) = I(1,0) \overset{\textbf{(I5)}}{=} 0$; **(GG3)** If $x{=}1$, then $GG_I(1,y) = I(N_I^{-1}(1),y) = I(0,y) \overset{\textbf{(I3)}}{=} 1$. On the other hand, if $y=1$, then $GG_I(x,1) = I(N_I^{-1}(x),1) \overset{\textbf{(I4)}}{=} 1$; **(GG4)** $x \le y \overset{\textbf{(N1)}}{\Rightarrow} N_I^{-1}(y) \le N_I^{-1}(x) \overset{\textbf{(I1)}}{\Rightarrow} I(N_I^{-1}(x),z) \le I(N_I^{-1}(y),z) \Rightarrow GG_I(x,z) \le GG_I(y,z)$; **(GG5)** It follows from the continuity of I and N_I^{-1}. So, GG_I is a GGF. □

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Lemma 3.2 aids to state the next result characterizing (GG,N)-implication functions.

Theorem 3.1. *For a function $I\colon [0,1]^2 \to [0,1]$, the following statements are equivalent:*

(i) $I{=}I_{GG,N}$ is a (GG,N)-implication function, N being strict and 0, the neutral element;
*(ii) I is continuous and satisfies **(I1)**, (NP) and L-CP(N_I^{-1}), where N_I is strict.*

Proof. $(i) \Rightarrow (ii)$ Let $I(x,y) = GG(N(x),y)$ where N is strict and 0 is neutral element of GG. So, by Prop. 3.1, I satisfies **(I1)**. From the continuity of GG and N, I is then continuous. Now, since 0 is a neutral element of GG, $\forall x \in [0,1]$, one has that $N_I(x) = I(x,0) = GG(N(x),0) = N(x)$. Thus, N_I is strict, since N is strict. Finally, by Props. 3.8(i) and 3.9(ii), $I_{GG,N}$ satisfies (NP) and L-CP(N_I^{-1}), respectively. $(ii) \Rightarrow (i)$ For $y \le z$, we have, by condition **(N2)**, that $N_I^{-1}(z) \le N_I^{-1}(y)$. So, since I satisfies **(I1)** and L-CP(N_I^{-1}), $\forall x \in [0,1]$, one has that $I(x,y) = I(N_I^{-1}(N_I(x)),y) \overset{\text{L-CP}}{=} I(N_I^{-1}(y),N_I(x)) \overset{\textbf{(I1)}}{\le} I(N_I^{-1}(z),N_I(x)) \overset{\text{L-CP}}{=} I(N_I^{-1}(N_I(x)),z) = I(x,z)$. Hence, I satisfies **(I2)**. Now, take $I(0,0) = I(N_I^{-1}(N_I(0)),0) \overset{\text{L-CP}}{=} I(N_I^{-1}(0),N_I(0)) = I(1,1) \overset{\textbf{(NP)}}{=} 1$. And $I(1,0) \overset{\textbf{(NP)}}{=} 0$. Thus, I satisfies **(I3*)**, **(I4*)** and **(I5)**, and I is a FIF. Also, by Lemma 3.2, $GG_I(x,y) = I(N_I^{-1}(x),y)$ is a GGF. So, $I_{GG_I,N_I}(x,y) = GG_I(N_I(x),y) = I(N_I^{-1}(N_I(x)),y) = I(x,y)$, $\forall x,y \in [0,1]$. Hence, I is a (GG,N)-implication function with $N = N_I$ being strict. So, by Prop. 3.3, 0 is a neutral element of GG_I. □

Finally, the next results follow immediately from Proposition 3.7 and Prop. 7.3.2 in [5].

Lemma 3.3. *Let $GG\colon [0,1]^2 \to [0,1]$ be a GGF and N be a strict fuzzy negation. If GG is associative, then it holds that $T(x,y) = N^{-1}(GG(N(x),N(y)))$ is a t-norm.*

Proposition 3.10. *Take a GGF $GG\colon [0,1]^2 \to [0,1]$, and let N be a strict fuzzy negation and $I_{GG,N}\colon [0,1]^2 \to [0,1]$ be a (GG,N)-implication function. If GG is associative, then, $I_{GG,N}$ satisfies (LI) with respect to a t-norm T if and only if $T(x,y) = N^{-1}(GG(N(x),N(y)))$.*

Proof. As GG is associative, by Prop. 3.7, GG is a t-conorm. So, it follows from Prop. 7.3.2 [5]. considering $T(x,y) = N^{-1}(GG(N(x),N(y)))$, by the Lemma 3.3, $T(x,y) = N^{-1}(GG(N(x),N(y)))$ is a t-norm. So, $I_{GG,N}(T(x,y),z) = GG(N(T(x,y)),z) = GG(N(N^{-1}(GG(N(x),N(y)))),z) = GG(GG(N(x),N(y)),z)$, now, by associativity of GG, $I_{GG,N}(T(x,y),z) = GG(N(x),GG(N(y),z))$ $= I_{GG,N}(x,I_{GG,N}(y,z))$. Therefore, $I_{GG,N}$ satisfies (LI) with respect to t-norm T. Conversely, if $I_{GG,N}$ satisfies (LI) with respect to a t-norm T, then $GG(N(T(x,y)),z) = GG(N(x),GG(N(y),z))$, for all $x,y,z \in [0,1]$. In particular, for $z = 0$, by associativity of GG, $GG(N(T(x,y)),0) = GG(N(x),GG(N(y),0)) = GG(GG(N(x),N(y)),0)$. Now, since 0 is the neutral element of GG, N(T(x,y)) = GG(N(x), N(y)). Therefore, $T(x,y) = N^{-1}(GG(N(x),N(y)))$, since N is strict. □

4 Conclusions

In this work, considering the contributions of GGF to several application areas, we focused on more flexible definitions, introducing a new and less restrictive class of fuzzy material implications, namely the (GG,N)-implication functions derived from GGF and fuzzy negations. Moreover, we studied several properties and provided their characterization. Ongoing work considers other implication functions derived from GOF and GGF, namely, the ones based on the residual, the quantum logic, and the Dishkant FIF and their intersections. Further work is also concerned with applications like (i) multi-criteria decision-making based on n-ary heterogeneous incomplete fuzzy preference relations, and (ii) fuzzy data stream clustering, which uses fuzzy dispersion and fuzzy similarity/dissimilarity obtained by the fuzzy material implications developed in this current work.

Acknowledgements

This work was partially funded by CAPES, FAPERGS/CNPq (23/2551-0001865-9) and CNPq (304118/2023-0, 407206/2023-0).

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