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# Competitive and Mutualism Model and Constraint Interval Theory

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**Abstract**. In this work we propose to study a competition/mutualism model considering the coefficients of interaction between species by constraint interval representation theory. The dynamics of the competition or mutualism can vary according to resources or population density. So, the interaction between species is defined by an interval coefficient, where the negative values mean cooperation, zero means no interaction and positive means competition.

**Keywords**. Competition, Mutualism, Interval Theory, Constraint Interval

## 1 Introduction

In models of ecology of population we have two important interactions between species. We can distinguished three types of interactions [2]: (a) Competition: two species are rivals in the exploitation of a commom resource; (b) Symbiosis, cooperation or mutualism: both species benefit from each other, for example algae and fungi; (c) Host-parasite: The parasites benefits from the host but they do it no good, for example the tapeworm a parasite of humans. There are different forms of competition [3]: categorized as a real competition are called interference and exploitation competition and apparent compettition when there is no interference between individuals. Many competitive models were studied and analyzed in relation the coexistence of species by changing the Lotka-Volterra model. Even in the case of a cooperative species, there may be a low level of competition between them due to similar resource use.

In general, there is an equilibrium level where the resources and space are sufficient for species to survive, but in the most societies we always have at least a lower level of competition between species. Actually, we have more challenges because the climate is changing and the survival will depend on our ability to cooperate or not. So, some small perturbations that caused little changes in the interactions on species are changing more and more. Zhang [9] quotes some references that the inter-specific competition can be reduced because of niche differentiation, spatially or temporally, colonization ability, moderate disturbance, aggregation enhancements, etc.

So, in [9] the author proposes competition/mutualism model by supposing that the interaction of one specie to the other is flexible instead of always negative. They considered that the zero growth isoclines is not a negative linear function to the population size of the competitor, but it is a parabolic function, that is, the mutualism happens at low density, but competition happens at high density.

Then, when we are studying a dynamic that is not static via mathematical modeling of competition/mutualism it is important that we consider the uncertainty in the coefficients that represent

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the interaction between multiple species. The probabilistic theory is one that is more usually considered in this cases. But in many cases it is not easy to have the enough precise data to a probability density function. Lately, some authors have studied problems with uncertainty via interval theory: in the evaluation and decision making [7], fault location in railroad [8], protein distance [6], covid-19 [1].

Here, we propose to study a similar idea as that in [9] considering the interaction coefficients between species are intervals, because changes occur in the interaction between them. Then, the idea is to study the competition/mutualism model considering the Constraint Interval(CI) theory. In the subsection 2.1 we recall some models and definition about interaction models between two species. Subsection 2.2 we recall definition about intervals numbers and eigenvalues via Constraint Interval (CI) theory, and competition/mutualism model is studied via CI.

# 2 Preliminaries

## 2.1 Models

In this subsection we recall some models of interaction between two species. The notation used in the first model is the same for the subsequent ones. The details can be found in [9].

**Definition 2.1.** *Consider the equation*

$$x' = f(x) \tag{1}$$

*defined on $U \subseteq \mathbb{R}^n$, then:*

1. *If $\dfrac{\partial f_i}{\partial x_j}(x) \leq 0, \forall x \in U$ and all $i \neq j$ then the population model is called cooperative model or we have a mutualism model;*

2. *If $\dfrac{\partial f_i}{\partial x_j}(x) \geq 0, \forall x \in U$ and all $i \neq j$ then the population model is called competitive model.*

Next follows the Lotka-Volterra competition, mutualism and the competition/mutualism models that are discussed in [9].

Consider the Lotka-Volterra competition model

$$\begin{cases} \dfrac{dx}{dt} &= \dfrac{r_1}{k_1} x\left(k_1 - x - \alpha y\right) \\ \\ \dfrac{dy}{dt} &= \dfrac{r_2}{k_2} y\left(k_2 - y - \beta x\right) \end{cases} \tag{2}$$

where $x, y$ are population numbers of species 1 and 2; $k_1, k_2$ are the carrying capacity of species 1 and 2; $r_1, r_2$ are the instantaneous rate for species 1 and 2, respectively; $\alpha$ and $\beta$ are the competition coefficient of species 1 and 2, respectively.

**Coexistence condition for the two species in (2):** It follows from the zero growth isoclines equations

$$\begin{cases} k_1 - x - \alpha y = 0 \\ k_2 - y - \beta x = 0 \end{cases} \tag{3}$$

that the equilibrium point $\left(\dfrac{k_1 - \alpha k_2}{1 - \alpha\beta}, \dfrac{k_2 - \beta k_1}{1 - \alpha\beta}\right)$ is stable if $x = k_1 - \alpha y > y = k_2 - \beta x$, then both species coexist.

 

The ecological implication of the mutualism model is that pure mutualism promotes carrying capacities of both species, and has high possibility of forming a stable equilibrium. Then, consider the model as it follows:

$$\begin{cases} \dfrac{dx}{dt} & = \dfrac{r_1}{k_1} x\,(k_1 - x + \alpha y) \\ \\ \dfrac{dy}{dt} & = \dfrac{r_2}{k_2} y\,(k_2 - y + \beta x) \end{cases}. \tag{4}$$

In [9] the author considers the interaction of one species to the other is flexible instead of always negative. Then, he proposed the competition/mutualism model as follows:

$$\begin{cases} \dfrac{dx}{dt} & = r_1 x \left(c_1 - x - a_1(y - b_1)^2\right) \\ \\ \dfrac{dy}{dt} & = r_2 y \left(c_2 - y - a_2(x - b_2)^2\right), \end{cases} \tag{5}$$

where $r_1, r_2, a_1, a_2, b_1, b_2, c_1, c_2$ are constants, and $r_1, r_2, c_1, c_2, a_1$ and $a_2$ are positive constants.

## 2.2 Constraint Interval Theory

In this subsection we recall some concepts about CI theory and CI matrix representation. The details can be found in [4] and [5]. Denote by $\mathbb{IR}$ the space of all real intervals.

The idea in Definition 2.2 below is to apply the mapping $\mathcal{R}$ to take each interval $[x] \in \mathbb{IR}$ (or $m-$tuple of intervals $([x_1], \ldots, [x_m]) \in \mathbb{IR} \times \ldots \times \mathbb{IR}$) for the space $\mathcal{F}$ (*or* $\mathcal{F} \times \ldots \times \mathcal{F}$) to one parameter (or $n \leq m$ parameters), $\mathcal{R}_{[x]}(\lambda_x)$ (*or* $(\mathcal{R}_{[x_1]}(\lambda_{x_1}), \ldots, \mathcal{R}_{[x_n]}(\lambda_{x_n}))$.

**Definition 2.2.** *A constraint interval representation (CI) is a representation of the interval* $[x] = [\underline{x}, \overline{x}]$, $[x] \in \mathbb{IR}$ *in the space* $\mathcal{F}$ *via a mapping* $\mathcal{R} : \mathbb{IR} \to \mathcal{F}$ *as follows*

$$\mathcal{R}([x]) = \mathcal{R}_{[x]} : [0,1] \to \mathbb{R}$$

*such that*

$$\mathcal{R}_{[x]}(\lambda_x) = \underline{x} + \lambda_x\,(\overline{x} - \underline{x},) = \underline{x} + \lambda_x w_x, \tag{6}$$

*for all* $\lambda_x \in [0,1]$*, where* $w_x = \overline{x} - \underline{x}$*. The representation space* $\mathcal{F}$ *is the space of bounded real-valued functions defined on* $[0,1]$.

**Definition 2.3.** *[5](Constraint Interval Extension) Given a continuous function* $f : U \subseteq \mathbb{R}^m \to \mathbb{R}$*, define* $\hat{f} : \mathbb{R}^m \to \mathbb{R}$ *by*

$$\hat{f}(x) = \begin{cases} f(x), & \textit{if } x \in U \\ 0, & \textit{otherwise} \end{cases}.$$

*Let* $\mathbb{U} \subset \mathbb{IR}^m$ *be given such that* $U \subseteq \mathbb{U}$*, for each* $[x] \in \mathbb{U}$*, let* $\phi_{f_{[x]}} : [0,1]^n \to \mathbb{R}$ *be given by*

$$\phi_{f_{[x]}}(\lambda_x) = (\hat{f} \circ \mathcal{R}_{[x]})(\lambda_x),$$

*where* $n \leq m$ *and* $\mathcal{R}_{[x]}$ *is the image of* $\mathcal{R} : \mathbb{U} \to \mathcal{F}^m$ *at* $[x]$*, that is,*

$$\mathcal{R}([x]) = \mathcal{R}_{[x]} : [0,1]^n \to \mathbb{R}^m.$$

*If* $\phi_{f_{[x]}} \in \mathcal{F}$*, then the interval function* $F^I : \mathbb{IU} \to \mathbb{IR}$ *given by*

$$F^I([x]) = \mathcal{MB}(\phi_{f_{[x]}})$$

*is called a* ***constraint interval function extension*** *of* $f$*. The map* $\Phi_f : \mathbb{U} \to \mathcal{F}$ *given by* $\Phi_f([x]) = \phi_{f_{[x]}}$ *is called a* ***constraint interval function representation of*** $F^I$.

 

**Remark 2.1.** *Given a matrix $A \in M_{m\times n}(\mathbb{R})$ we can write the elements of matrix $A$ as $(a_{11},\dots,a_{1n},\dots,a_{m1},\dots,a_{mn}) \in \mathbb{R}^{n\times m}$ . We denote the elements in $\mathbb{R}^{n\times m}$ that are independent variables of the function $\mathcal{R}_{[A]} : [0,1]^{m\times n} \to \mathbb{R}^{m_1\times n_1}, m \le m_1$ and $n \le n_1$.*

**Definition 2.4.** *Let $[A] = [\underline{A} \quad \overline{A}]$ be an interval matrix, then the $CI$matrix is defined by*

$$\mathcal{R}_{[A]}(\Gamma) = \begin{pmatrix} \underline{a}_{11} + \gamma_{a_{11}} w_{\mathbf{a}_{11}} & \dots & \underline{a}_{1n} + \gamma_{a_{1n}} w_{\mathbf{a}_{1n}} \\ \dots & \dots & \dots \\ \underline{a}_{m1} + \gamma_{a_{m1}} w_{\mathbf{a}_{m1}} & \dots & \underline{a}_{mn} + \gamma_{a_{mn}} w_{\mathbf{a}_{mn}} \end{pmatrix} = \underline{A} + \Gamma_A \bigodot W_A$$

*where $\Gamma_A = (\gamma_{a_{ij}}), \underline{A} = (\underline{a}_{ij}), 0 \le \gamma_{a_{ij}} \le 1,$ for $i = 1,\dots,m$ and $j = 1,\dots,n$, $\bigodot$ is the componentwise product between the matrices $\Gamma_A = (\gamma_{a_{ij}})$ and $W_A = \left(\overline{a}_{ij} - \underline{a}_{ij}\right), \Gamma = (\gamma_{a_{11}},\dots,\gamma_{a_{1n}},\dots,\gamma_{a_{m1}},\dots,\gamma_{a_{mn}}) \in \mathbb{R}^{m\times n}$. Then, if $m = n$ we say that $\lambda^*(\Gamma)$ is an eigenvalue of $\mathcal{R}_{[A]}(\Gamma)$ if $\exists\, v \ne 0$ vector $\mid \mathcal{R}_{[A]}(\Gamma)v = \lambda^*(\Gamma)v$. that is,*

$$det(\mathcal{R}_{[A]}(\Gamma) - \lambda^*(\Gamma) \times I_n) = 0, \tag{7}$$

*for some choice of the parameters matrix $\Gamma_A, A \in [A]$, where $I_n$ is the identity matrix of order $n$.*

Considering the CI matrix, if we have a CI differential autonomous system $X'(t) = A(\Gamma)X$, then for each parameter matrix fixed, we have a standard differential autonomous system of type $X'(t) = AX$, where $A \in M_n(\mathbb{R})$ is a real matrix.

Then, if in problem (2), the parameters $\alpha$ and $\beta$ of competition have uncertainty such that $\alpha \in [\alpha] = [\underline{\alpha}, \overline{\alpha}]$ and $\beta \in [\beta] = [\underline{\beta}, \overline{\beta}]$, then applying the mapping $\mathcal{R}$ in both intervals, we have $\mathcal{R}_{[\alpha]}(\gamma_\alpha) = \underline{\alpha} - \gamma_\alpha(\overline{\alpha} - \underline{\alpha})$ and $\mathcal{R}_{[\beta]}(\gamma_\beta) = \underline{\beta} - \gamma_\beta\left(\overline{\beta} - \underline{\beta}\right)$ with $\gamma_\alpha, \gamma_\beta \in [0,1]$. So, it follows that

$$\begin{cases} \dfrac{dx}{dt} &= r_1 x \left(\dfrac{k_1 - x - (\underline{\alpha} - \gamma_\alpha w_\alpha) y}{k_1}\right) \\ \\ \dfrac{dy}{dt} &= r_2 y \left(\dfrac{k_2 - y - (\underline{\beta} - \gamma_\beta w_\beta) x}{k_2}\right), \end{cases} \tag{8}$$

where $w_\alpha = \overline{\alpha} - \underline{\alpha}$ and $w_\beta = \overline{\beta} - \underline{\beta}$.

Observe that the type of interaction between the species $x$ and $y$ in the model (8) will depend on which interval we are considering the parameters $\alpha$ and $\beta$, because calling $f_1(x,y) = r_1 x \left(\dfrac{k_1 - x - I_1(\gamma_\alpha) y}{k_1}\right)$ and $f_2(x,y)) = r_1 y \left(\dfrac{k_2 - y - I_2(\gamma_\beta) x}{k_2}\right)$, where $I_1(\gamma_\alpha) = \underline{\alpha} + \gamma_\alpha w_\alpha$ and $I_1(\gamma_\beta) = \underline{\beta} - \gamma_\beta w_\beta$. Then,

$$\begin{cases} \dfrac{\partial f_1}{\partial y} &= -\dfrac{r_1}{k_1} x I_1(\gamma_\alpha) \\ \dfrac{\partial f_2}{\partial x} &= -\dfrac{r_2}{k_2} y(\gamma_\beta), \end{cases} \tag{9}$$

where $r_i > 0, k_i > 0, x \ge 0$ and $y \ge 0$ for $i = 1, 2$.

Thus, by Definition 2.1 and (9) it follows that:

1. If $\overline{\alpha} < 0$ and $\overline{\beta} < 0$, then the system is a competitive model;
2. If $\underline{\alpha} > 0$ and $\underline{\beta} > 0$ then the system is a mutualism model. Here, the negative signs in the parameters mean that there are no more any competition between the species because for survival there is no necessity to fight;
3. If $\underline{\alpha} = \overline{\alpha}$ and $\underline{\beta} = \overline{\beta}$, then no uncertainty about the type of model. But, if both are equal zero, then there is no interaction between the species.

 

4. If $\underline{\alpha}, \underline{\beta} < 0$ and $\overline{\alpha}, \overline{\beta} > 0$, then the interaction between the species are changing from mutualism for competition or the opposite. For example, if two countries have the enough resources to survive it is possible for them to live without competition, but if one country doesn't have for example enough food we can have a war between them.

The fixed points are: $(0,0),(0,k_2),(k_1,0)$ and

$$\left(\frac{k_1 - k_2 I_1(\alpha)}{1 - I_1(\gamma_\alpha)(I_2(\gamma_\beta))}, \frac{k_2 - I_2(\gamma_\beta)k_1}{1 - I_1(\gamma_\alpha)(I_2(\gamma_\beta))}\right).$$

So, the stability will be defined considering the Jacobian matrix

$$J(x,y) = \begin{bmatrix} \frac{r_1}{k_1}\left(k_1 - 2x - I_1(\gamma_\alpha)y\right) & -\frac{r_1}{k_1} I_1(\gamma_\alpha)x \\ -\frac{r_2}{k_2} I_2(\gamma_\beta)y & \frac{r_2}{k_2}\left(k_2 - 2y - I_2(\gamma_\beta)y\right) \end{bmatrix} \tag{10}$$

Note that the classification of fixed points will depend on $I_1(\gamma_\alpha)$ and $I_2(\gamma_\beta)$ except for the point $(0,0)$. For the better understanding we will study a particular case.

**Example 2.1.** *Consider the model*

$$\begin{cases} x'(t) & = x(3 - x + \alpha y) \\ y'(t) & = y(2 - y + \beta x) \end{cases} \tag{11}$$

*Now, if there are changes in the interactions between two species because weather, source of food, etc, take $\alpha \in [-0.5, 2]$ and $\beta \in [-1, 1]$, then applying the mapping $\mathcal{R}$ in both intervals, we have $\mathcal{R}_{[-0.5,2]}(\gamma_\alpha) = -0.5 + 2.5\gamma_\alpha$ and $\mathcal{R}_{[-1,1]}(\gamma_\beta) = -1 + 2\gamma_\beta$, with $\gamma_\alpha, \gamma_\beta \in [0,1]$. Thus, the system (11) is modeled as*

$$\begin{cases} x'(t) & = x(3 - x + (-0.5 + 2.5\gamma_\alpha)y) \\ y'(t) & = y(2 - y + (-1 + 2\gamma_\beta)x) \end{cases} \tag{12}$$

*where $\gamma_\alpha, \gamma_\beta \in [0,1]$.*

*The fixed points for (12) are: $(0,0),(0,2),(3,0)$ and*

$$\left(\frac{3 + 2(-0.5 + 2.5\gamma_\alpha)}{1 - (-0.5 + 2.5\gamma_\alpha)(-1 + 2\gamma_\beta)}, \frac{2 + 3(-1 + 2\gamma_\beta)}{1 - (-0.5 + 2.5\gamma_\alpha)(-1 + 2\gamma_\beta)}\right) = (x^*, y^*).$$

*Then, the Jacobian matrix is:*

$$J(x,y) = \begin{bmatrix} 3 - 2x + (-0.5 + 2.5\gamma_\alpha)y & (-0.5 + 2.5\gamma_\alpha)x \\ (-1 + 2\gamma_\beta)y & 2 - 2y + (-1 + 2\gamma_\beta)x. \end{bmatrix} \tag{13}$$

*such that $det(J(x,y) - \lambda I_2) = 0$, evaluated in the fixed points:*

1. *At $(0,0), \lambda^2 - 5\lambda + 6 = 0$ the eigenvalues are $3, 1$ and the equilibruim point is unstable.*

2. *At $(0,2), \lambda^2 - (4 - 5\gamma_\alpha - 2)\lambda - 2(4 - 5\gamma_\alpha) = 0$, then the eigenvalues are $\lambda_1 = -4$ and $\lambda_2(\gamma_\alpha) = 4 - 5\gamma_\alpha, \gamma_\alpha \in [0,1]$ such which $\lambda_1 = -4$ and the interval eigenvalue is $[\lambda_2] = [-1, 4]$, generating stable or unstable behavior in this point.*

3. *At $(3,0)$ the eigenvalues are $\lambda_1 = -3$ and $\lambda_2(\gamma_\beta) = -1 + 6\gamma_\beta, \gamma_\beta \in [0,1]$ such that $\lambda_1 = -3$ and the interval eigenvalue $[\lambda_2] = [-1, 5]$, generating stable or unstable behavior in this point.*

 

*4. At $(x^*, y^*)$ the eigenvalues are*

$$\lambda_1(\Gamma) = \frac{-2.5 - 12.5\gamma_\alpha - \sqrt{(-1.5 + 2.5\gamma_\alpha)^2 + 4(-0.5 + 2.5\gamma_\alpha)(-1 + 2.5\gamma_\beta)(2 + 5\gamma_\alpha)(-1 + 6\gamma_\beta)}}{-2 + 2(-.5 + 2.5\gamma_\alpha)(-1 + 2\gamma_\beta)}$$

*and*

$$\lambda_2(\Gamma) = \frac{-2.5 - 12.5\gamma_\alpha + \sqrt{(-1.5 + 2.5\gamma_\alpha)^2 + 4(-0.5 + 2.5\gamma_\alpha)(-1 + 2.5\gamma_\beta)(2 + 5\gamma_\alpha)(-1 + 6\gamma_\beta)}}{-2 + 2(-.5 + 2.5\gamma_\alpha)(-1 + 2\gamma_\beta)}$$

*where $\Gamma = (\gamma_\alpha, \gamma_\beta)$.*

*Maximizing and minimizing $\lambda_1(\Gamma)$ and $\lambda_2(\Gamma)$ subject to $0 \leq \gamma_\alpha \leq 1, 0 \leq \gamma_\beta \leq 1$ we have the following interval eigenvalues:*

$[\lambda_1] = [\lambda_1(0.918621, 0.831757) = -73.6585, \lambda_1(0.858006, 0.555602) = 11.8647]$ *and*

$[\lambda_2] = [\lambda_2(0.918621, 0.831757) = 0.846459, \lambda_2(0.858006, 0.555602) = 4.32132]$.

*So, in this case we can have coexistence for both negative eigenvalues, but if both are positive or one of them is negative the fixed point is unstable. Below in the Figure 1 we have the behavior according two scenarios for (12): (a) There is competition between the two species, so for $\gamma_\alpha = \gamma_\beta = 0$, the populations $x$ have increased until stabilization and the population $y$ has decreased; (b) There is no competition, so for $\gamma_\alpha = 0.6$ and $\gamma_\beta = 0.75$, the populations have increased until stabilization.*

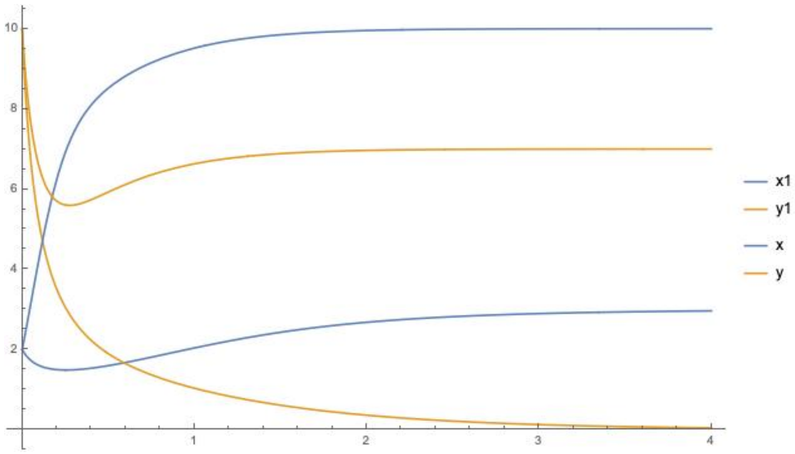


Figure 1: Solution of the system with the initial conditions $x(0) = 0, y(0) = 0$ and parameters $\gamma_\alpha = \gamma_\beta = 0$ and $\gamma_\alpha = 0.6, \gamma_\beta = 0.75$ for (12). Fonte: author.

# 3 Final Considerations

We proposed a mutualism/competition model using the constraint interval eigenvalue representation. So we can analyze the changes in the interactions between species considering that a competition can stop and start a cooperation when the sources are enough for both and another time the cooperation can be finish if the sources are not enough for the populations. The model presented here corroborates with the analysis of the authors in [3], where they say that some mechanism avoid competition exclusive, reducing competitive interactions to increase the strategies for new colonization and nutrient aquisition.

 

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