Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 11, n. 1, 2025.

Trabalho apresentado no XLIII CNMAC, Centro de Convenções do Armação Resort - Porto de Galinhas - PE, 2024

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Some Lagrangian Calculations for Isotropic Random Fields

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Abstract. This work presents a general form for a two-dimensional, mean-zero, homogeneous, stationary, incompressible, and isotropic random velocity field which is written as a sum of many Fourier modes. We assume that the Lagrangian auto-covariance function can be written as a Taylor series and then we develop a methodology to evaluate such Taylor coefficients. Additionally, we identify a notable pattern in terms associated with the highest moments of the random magnitudes for each order. We analyze the convergence of the Taylor series formed only by those terms.

Keys words. Isotropic fields, Fourier modes, Taylor expansion, passive trace transport

1 Introduction

An important problem in statistical fluid mechanics is to obtain the statistical descriptions of the motion of a single particle in a random velocity field. Considering the passive tracer transport problem as a simpler case, which involves determining the law of the entire stochastic location process $\mathbf{X}_t$, $t \geq 0$, of a single particle at time $t \geq 0$, which is moved by a random velocity field $\mathbf{U}$ when the motion of the particle itself does not affect the random velocity field. Even for this problem, given the law of the random velocity field $\mathbf{U}$, we still have limited ability to produce theoretical general results for the location process $\mathbf{X}_t$. For references, see [1, 2, 6, 7]. In other words, we can formulate the passive tracer problem as a stochastic initial value problem as below.

Let $\mathbf{U} = \{\mathbf{U}(\mathbf{x}, t), \mathbf{x} \in \mathbb{R}^2, t \geq 0\}$ be a random velocity field, and let $\mathbf{X}_t$ be the particle position at time $t \geq 0$. So, $\mathbf{X}_t, t \geq 0$, is the solution of the differential equation of the motion given by

$$\frac{d\mathbf{X}_t}{dt} = \mathbf{U}(\mathbf{X}_t, t), \quad t > 0; \quad \mathbf{X}_0 = \mathbf{0}. \tag{1}$$

The main goal is determining the law of the entire stochastic location process $\mathbf{X}_t, t \geq 0$, given the law of the random velocity field $\mathbf{U}$.

Closely related to the passive tracer problem is the task of determining the law of the Lagrangian velocity process $\mathbf{U}(\mathbf{X}_t, t)$, $t \geq 0$, which represents the particle's velocity as observed by an observer whose location $\mathbf{X}_t$ is determined by the environment. The Eulerian description provided by $\mathbf{U}(\mathbf{x}, t)$, where the coordinate system is fixed, differs from the Lagrangian description $\mathbf{U}(\mathbf{X}_t, t)$, which offers a description of the velocity field from the perspective of a particle following the velocity field.

For this article we take a two-dimensional, mean-zero, homogeneous, stationary, and incompressible velocity field written as a sum of finitely many Fourier modes as

$$\mathbf{U}(\mathbf{x}, t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} R_n \sin(\mathbf{W}_n \cdot \mathbf{x} + \Phi_n)\, \mathbf{\Theta}_n, \quad \mathbf{x} \in \mathbb{R}^2, \tag{2}$$

with $\mathbf{\Theta}_n = \mathbf{W}_n^{\perp} = [-W_{n,2}, W_{n,1}]^T$, where $\mathbf{W}_n = [W_{n,1}, W_{n,2}]^T$, for $n = 1, 2, \ldots, N$, and random amplitudes R_n and random wave numbers $\mathbf{W}_n$ are independent of the random phases Φ_n, in the

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sense that the collection $(R_1, \mathbf{W}_1, R_2, \mathbf{W}_2, \cdots, R_N, \mathbf{W}_N)$ is independent of $(\Phi_1, \Phi_2, \ldots, \Phi_N)$. In addition, we assume that the random phases Φ_n, $n = 1, 2, \ldots, N$, are independent and uniformly distributed on $[0, 2\pi]$, and random vectors $(R_n, \mathbf{W}_n)$, $n = 1, 2, \ldots, N$ have finite joint moments.

Remark 1.1. *In previous works [3, 5], we used trigonometric velocity fields, as in Eq. (2), to derive statistical properties of such random fields. Specifically, we constructed Gaussian random fields and presented numerical evidence that the joint distribution of* $(\mathbf{U}(\mathbf{0}, 0), \mathbf{U}(\mathbf{X}_t, t))$*, for each* $t > 0$*, is not Gaussian, even for such Gaussian fields. Additionally, we obtained the first terms of the Taylor expansion for the Lagrangian auto-covariance function, which is an important statistical characteristic of the Lagrangian velocity process* $\mathbf{U}(\mathbf{X}_t, t)$*,* $t \geq 0$*.*

For this work, we consider more specialized random fields, namely isotropic fields, by assuming an additional hypothesis upon the form of random wave numbers $\mathbf{W}_n$, $n = 1, 2, \ldots, N$. Isotropic fields have no preferred direction in $\mathbf{U}(\mathbf{x}, t)$ or, to be more precise, the statistical distribution of $\mathbf{U}(\mathbf{x}, t)$ is not affected by all possible rotations passing through the origin. This assumption allows us to derive additional theoretical results and make conjectures about the convergence of the Taylor expansion for the Lagrangian auto-covariance function.

2 Isotropic Random Fields

Suppose that each random wave number W_n, for $n = 1, 2, \ldots, N$, can be written as a product of its magnitude times a vector living in the unit circle, that is,

$$W_n = M_n \begin{bmatrix} \cos(\Psi_n) \\ \sin(\Psi_n) \end{bmatrix}, \quad n = 1, 2, \ldots, N, \tag{3}$$

where $M_n = \|W_n\|$, for $n = 1, 2, \ldots, N$, is the random magnitude, and Ψ_n, for $n = 1, 2, \ldots, N$, is the random wave number angle. Moreover, assume that random vector (R_n, M_n) is independent of Ψ_n, for all $n = 1, 2, \ldots, N$, and $(\Psi_1, \Psi_2, \ldots, \Psi_N)$ is a collection of independent and uniformly distributed random variables on $[0, 2\pi]$.

Remark 2.1. *Notice that we still allow some dependence between random amplitudes* R_n *and random magnitude* M_n*, for* $n = 1, 2, \ldots, N$*. But both random variables* R_n *and* M_n *are independent of the wave number angle* Ψ_n*, for* $n = 1, 2, \ldots, N$*.*

Theorem 2.1. *Let* $\mathbf{U}(\mathbf{x}, t)$ *be a velocity field as defined in Eq.(2). Additionally, assume that each random wave number is as defined in Eq.(3). Then, the velocity field* $\mathbf{U}(\mathbf{x}, t)$ *is isotropic, meaning that the covariation matrix* $Cov(\mathbf{U}(\mathbf{x}, t), \mathbf{U}(\mathbf{y}, t))$ *depends on* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ *only through* $\|\mathbf{x} - \mathbf{y}\|$*, which does not depend on angles.*

Proof. For details, see [4]. □

3 The Lagrangian Auto-Covariance Function

Let us start with the definition and some properties of the Lagrangian auto-covariation function.

Definition 3.1. *Let* $s', t' \geq 0$ *be real numbers. We define the Lagrangian auto-covariance of* $\mathbf{U}$ *by*

$$\Sigma_L(s', t') = \mathbb{E}\left[\mathbf{U}(\mathbf{X}_{\mathbf{s}'}, s')\mathbf{U}(\mathbf{X}_{\mathbf{t}'}, t')^T\right], \tag{4}$$

where $\mathbf{X}_t$ *satisfies the equation of the motion and the initial condition according to Eq.*(1)*.*

Remark 3.1. *Let* $\mathbf{U}(\mathbf{x}, t)$ *be defined as in Eq. (2). Then the stochastic process* $\mathbf{U}(\mathbf{X}_t, t)$*,* $t \geq 0$*, is stationary. Consequently, the Lagrangian auto-covariance depends only on the difference* $t = t' - s'$*, allowing us to express the Lagrangian auto-covariance as a function of* $t \geq 0$ *as* $\mathbf{\Sigma}_L(s', t') = \mathbf{\Sigma}_L(t)$*.*

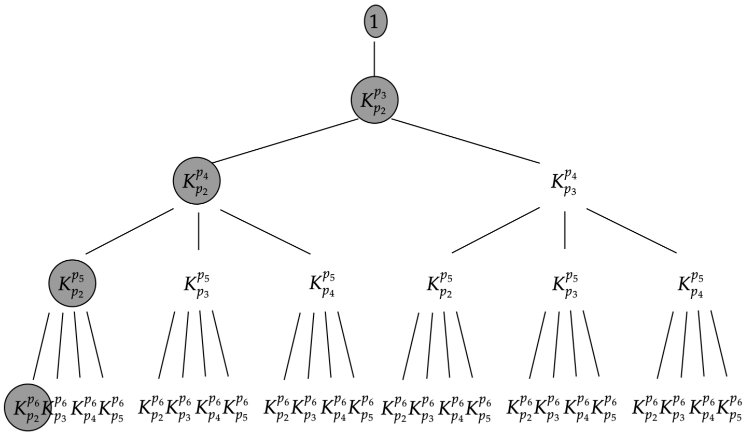

Figure 1: Tree diagram T with branches up to order 4. Source: Author.

Remark 3.2. *Taylor coefficients $C^{(m)}$ for the Lagrangian auto-correlation function are given by*

$$C^{(m)} = \frac{1}{m!}\frac{1}{N^{m/2+1}}\sum_{p\in\mathcal{P}}\sum_{i\in T}\mathbb{E}\left[\mathbf{G}_i^p(R,\mathbf{W})\right]\mathbb{E}\left[F_i^p(S,C)\right], \quad m=0,1,\ldots, \tag{5}$$

where $p=(p_1,p_2,\ldots,p_{m+2})\in\mathcal{P}$ with $\mathcal{P}=\{1,2,\ldots,N\}^{m+2}$, $i=(i_3,\ldots,i_{m+2})$ is a multi-index which ranges over the routes from the root of the tree T to the leaves (see Figure 1), K_i is the product of factors along branch i, $\mathbf{G}_i^p(R,\mathbf{W}) = R_{p_1}\cdots R_{p_{m+2}}\mathbf{\Theta}_{p_1}\mathbf{\Theta}_{p_2}^T K_i$ is a function of random amplitudes and wave numbers, and $F_i^p(S,C) = S_{p_1}D^i(S_{p_2}\cdots S_{p_{m+1}})S_{p_{m+2}}$ is a function of random phases, for each $i\in T$. Since we are evaluating at $t=0$, the abbreviation S_{p_j} now equals $\sin(\Phi_{p_j})$ and C_{p_j} equals $\cos(\Phi_{p_j})$. For the case when we assume that random vectors (R_n,W_n), for $n=1,2,\ldots,N$, are identically distributed and the number of Fourier modes $N\to\infty$, then Eq.(5) simplifies to

$$C^{(m)} = \frac{1}{m!}\sum_{i\in T}\sum_{p\in\mathfrak{D}}\mathbb{E}\left[\mathbf{G}_i^p(R,\mathbf{W})\right]\mathbb{E}\left[F_i^p(S,C)\right], \tag{6}$$

which does not depend on the number of Fourier modes N. For details, see [4].

Notice factors $\mathbb{E}\left[\mathbf{G}_i^p(R,\mathbf{W})\right]$ as in Eq.(6) depend on random wave-numbers $W_n, n=1,2,\ldots,N$. More explicitly, $\mathbb{E}\left[\mathbf{G}_i^p(R,\mathbf{W})\right] = \mathbb{E}\left[R_{p_1}R_{p_2}\cdots R_{p_{m+2}}\mathbf{\Theta}_{p_1}\mathbf{\Theta}_{p_2}^T K_{p_{i_1}}^{p_3}K_{p_{i_2}}^{p_4}\cdots K_{p_{i_m}}^{p_{m+2}}\right], i\in T$. Moreover, we can express the matrix product $\mathbf{\Theta}_{p_1}\mathbf{\Theta}_{p_2}^T$ and each factor $K_r^s, r,s=1,2,\ldots,N$, using random magnitudes and random wave number angles as in Eq.(3), to get

$$\mathbb{E}\left[\mathbf{G}_i^p(R,\mathbf{W})\right] = \mathbb{E}\left[\mathbf{R}M_{p_1}M_{p_2}\mathbf{M}\right]\mathbb{E}\left[\begin{bmatrix}\sin(\Psi_{p_1})\sin(\Psi_{p_2}) & -\sin(\Psi_{p_1})\cos(\Psi_{p_2})\\ -\cos(\Psi_{p_1})\sin(\Psi_{p_2}) & \cos(\Psi_{p_1})\cos(\Psi_{p_2})\end{bmatrix}\mathbf{T}\right], \tag{7}$$

which is a squared 2×2 matrix, where $\mathbf{R} = R_{p_1}R_{p_2}\cdots R_{p_{m+2}}$ is a product of random amplitudes, $\mathbf{M} = M_{p_{i_1}}M_{p_3}M_{p_{i_2}}M_{p_4}\cdots M_{p_{i_m}}M_{p_{m+2}}$ is the product of random magnitudes from each factor $K_{p_{i_r}}^{p_{r+2}}$, for $r=1,2,\ldots,m$, and $\mathbf{T}=T_1T_2\cdots T_m = \sin(\Psi_{p_{i_1}}^{p_3})\sin(\Psi_{p_{i_2}}^{p_4})\cdots\sin(\Psi_{p_{i_m}}^{p_{m+2}})$ is the product of sines of the angle between wave numbers $W_{p_{i_r}}$ and $W_{p_{r+2}}$, for $r=1,2,\ldots,m$, $i\in T$ and $p\in\mathfrak{D}$.

Theorem 3.1. *Let $m>0$ be an even number. Then each factor $\mathbb{E}\left[\mathbf{G}_i^p(R,\mathbf{W})\right]$ as in Eq.(7) is a diagonal squared matrix, and so is the Lagrangian auto-covariance function.*

Proof. For details, see [4]. □

Remark 3.3. *Notice that we only need moments involving even powers of the random amplitudes R_n and random magnitudes M_n, for $n = 1, 2, \ldots, m/2+1$, to evaluate expressions as in Eq.(7). In fact, as $p \in \mathfrak{D}$, then factors as $R_{p_1} R_{p_2} \cdots R_{p_m+2}$ become $R_{p_1}^2 R_{p_2}^2 \cdots R_{p_{m/2+1}}^2$.*

Remark 3.4. *Notice that, given $i \in T$ and $p \in \mathfrak{D}$, we can evaluate the numerical value of factors*

$$\mathbb{E}\left[\Upsilon_{p_1}^{p_2}\mathbf{T}\right] = \mathbb{E}\left[\begin{bmatrix} \sin(\Psi_{p_1})\sin(\Psi_{p_2}) & -\sin(\Psi_{p_1})\cos(\Psi_{p_2}) \\ -\cos(\Psi_{p_1})\sin(\Psi_{p_2}) & \cos(\Psi_{p_1})\cos(\Psi_{p_2}) \end{bmatrix}\mathbf{T}\right], \tag{8}$$

in Eq.(7), since Ψ_n, for $n = 1, 2, \ldots, m/2+1$, are independent and uniformly distributed on $[0, 2\pi]$.

3.1 Symbolic Expressions for Taylor Coefficients

We developed and implemented a computational procedure to obtain symbolic expressions for higher-order derivatives of the Lagrangian auto-correlation function evaluated at $t = 0$, see [5]. Ultimately, this procedure yields the Taylor coefficients as described in Eq.(6). In fact, we obtained symbolic expressions for terms such as $\mathbb{E}\left[\mathbf{G}_i^p(R, \mathbf{W})\right]\mathbb{E}\left[F_i^p(S, C)\right]$. Moreover, notice that the sum on the right-hand side of Eq.(5) has $(m+1)(m-1)\cdots 1 \cdot m!$ terms, which does not depend on N.

Assuming that the random variables $(R_1, \mathbf{W}_1), \ldots, (R_N, \mathbf{W}_N)$ are independent and identically distributed. This allows us to recognize that as we sum over $p_1, \ldots, p_{m+2}$ going from 1 to N, many of the terms have the same numerical values.

Example 3.1. *Using Eq. (6) to evaluate the fourth-order Taylor coefficient, we need to collect* 360 *terms since T has* 24 *elements and $\mathfrak{D}$ has* 15 *elements. However, many of these terms are* 0*. In fact, after listing all non-zero terms, we can relabel some indices and sort factors to group similar terms. Finally, we end up with a list having only* 4 *distinct terms, as shown in Figure 2.*

```
         Symbolic expressions for distinct terms for m=4:
line 1:  0 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^3)(K_2^3)(K_2^3)(K_2^3)] * E[S1^2]^3
line 2:  1 * E[R1^2 R2^2 R3^2 * Theta_1^1 * (K_1^2)(K_1^2)(K_2^3)(K_2^3)] * E[S1^2]^3
line 3:  3 * E[R1^2 R2^2 R3^2 * Theta_1^1 * (K_1^2)(K_1^2)(K_1^3)(K_1^3)] * E[S1^2]^3
line 4: -5 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_1^2)(K_1^3)(K_2^3)] * E[S1^2]^3
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Figure 2: Computer output of distinct terms for $m = 4$. Source: Author.

Remark 3.5. *The symbolic expressions in Figure 2 depend on the joint distribution of the random variables $(R_n, \mathbf{W}_n)$ and the numerical value of $\mathbb{E}\left[S_1^2\right]$. To calculate numerically, recognize that the $Theta_*^*$ factors and K_*^* depend on wave numbers $\mathbf{W}_n$. One can multiply these out, use linearity to separate out terms, then use independence of the $(R_n, \mathbf{W}_n)$ to get a product of expected values. This reduces the computation to joint moments of R_n and $\mathbf{W}_n$, $n = 1, 2, \ldots, m/2+1$. The highest moment encountered will be $\mathbb{E}\left[R_1^2|\mathbf{W}_1|^{m+2}\right]$. Therefore, the methodology we use to obtain Taylor coefficients for the Lagrangian auto-covariance is quite general and allows us to explore a large variety of scenarios by changing the joint distribution of random amplitudes and wave numbers.*

Assuming the isotropic condition, described as in Eq.(3), we can split expectations involving random amplitudes R_n and random magnitudes M_n from random wave number angles Ψ_n, for $n = 1, 2, \ldots, m/2+1$, according to Eq.(7). Moreover, Theorem 3.1 tells us $\mathbb{E}\left[\mathbf{G}_i^p(R, \mathbf{W})\right]$ is a diagonal matrix. Hence, it remains to determine explicitly an expression for the main diagonal terms in order to obtain an expression for Taylor coefficients of the Lagrangian auto-covariance. In fact, the wave number angles Ψ_n, for $n = 1, 2, \ldots, m/2+1$, are independent and uniformly distributed on $[0, 2\pi]$, from which we can conclude that the main diagonal terms are all equal.

Remark 3.6. *In expressions as in Figure 3, we can determine expectations involving random wave number angles and random phases since these variables are independent and uniformly distributed on $[0, 2\pi]$. So we can evaluate numerically the last two expectations factors in each expression.*


```
        Symbolic expressions for non-null distinct terms for m=4:
line 1:  1 * E[R1^2 R2^2 R3^2 * M1^4 M2^4 M3^2] * E[S1 S1 (S_1^2)(S_1^2)(S_2^3)(S_2^3)] * E[S1^2]^ 3
line 2:  3 * E[R1^2 R2^2 R3^2 * M1^6 M2^2 M3^2] * E[S1 S1 (S_1^2)(S_1^2)(S_1^3)(S_1^3)] * E[S1^2]^ 3
line 3: -5 * E[R1^2 R2^2 R3^2 * M1^4 M2^4 M3^2] * E[S1 S1 (S_1^2)(S_1^2)(S_1^3)(S_2^3)] * E[S1^2]^ 3
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Figure 3: Computer output of distinct terms for $m = 4$ for the isotropic case. Source: Author.

Remark 3.7. *Terms in lines* 1 *and* 3 *in Figure 3 have the same factor involving random amplitudes and magnitudes. Moreover, according to Remark 3.6, we can evaluate the last two expectations to obtain the terms as in Figure 4. Therefore, in order to evaluate the fourth-order Taylor coefficient, we need to determine two more distinct expectations involving random amplitudes and magnitudes.*

```
         (a) Symbolic expressions for non-null distinct terms for m=4:          (b) Symbolic expressions for non-null distinct terms for m=4:
line 1:   -0.03125000 * E[R1^2 R2^2 R3^2  * M1^4 M2^4 M3^2]                ->   -0.03125000 * E[M1^4] E[M2^4] E[M3^2]
line 2:    0.37500000 * E[R1^2 R2^2 R3^2  * M1^6 M2^2 M3^2]                ->    0.37500000 * E[M1^6] E[M2^2] E[M3^2]
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```

Figure 4: Computer output for terms depending on random amplitudes and magnitudes. Source: Author.

Remark 3.8. *Notice that we can assign different distributions to the random variables of the model, which allows us to obtain numerically the Taylor coefficients without any simulations. This illustrates how robust this methodology is. So we can calculate the Lagrangian auto-correlation for distinct setups quickly, which ultimately ends up being far faster than Monte Carlo simulations.*

Assuming that random amplitudes R_n and random magnitudes M_n are independent allows us to split expectations in the expressions shown in Figure 4(a), which initially can depend on the joint distribution of $(R_1, M_1, \ldots, R_{m/2+1}, M_{m/2+1})$. Additionally, assuming that random amplitudes are deterministic and equal to 1, that is, $\mathbb{P}(R_n = 1) = 1$, for $n = 1, 2, \ldots, m/2 + 1$, and wave number magnitudes M_n are independent and identically distributed for $n = 1, 2, \ldots, m/2 + 1$, then such expressions depend only on even moments of the random magnitude M_n, for $n = 1, 2, \ldots, m/2+1$, as shown in Figure 4(b). We are able to evaluate the Taylor coefficients of the Lagrangian auto-correlation numerically once we properly assign the distribution of the random magnitude M_1.

Remark 3.9. *In Figure Figure 4(b), notice that the expressions represent the terms of the fourth-order derivative of the Lagrangian auto-correlation at* $t = 0$*. To obtain the Taylor coefficient of the fourth-order, we need to add these two terms together and then multiply by* $1/4!$*, as indicated in Eq.(5). Figure 5 presents a complete list of terms for the Taylor expansion up to order* $m = 10$*. We also highlight the highest moment of the random amplitude* M_1 *for each term in this list.*

Example 3.2. *By assigning different distributions for the random magnitude* M_1*, we can evaluate many moments of* M_1 *and then assemble approximations for the Taylor expansion and analyze its behavior for each distribution. In Figure 6, for example, we set two different distributions for* M_1*: (a) where the random magnitude* M_1 *is constant and equal to 1, and (b) where* M_1 *is uniformly distributed on* $[0, 2]$*, which means that* M_1 *has mean 1 in both cases.*

Remark 3.10. *Examining the terms with the highest moments of the random magnitude* M_1*, for each order* m *in Figure 5, we observe the coefficients are* $c_0 = 0.5$*,* $c_2 = -0.25 = -0.5 \cdot 0.5$*, …, and* $c_{10} = -14.765625 = -0.5 \cdot 0.5 \cdot 1.5 \cdot 2.5 \cdot 3.5 \cdot 4.5$*. This observation allows us to conjecture that*

$$a_m = (-1)^{m/2} 0.5 \prod_{j=1}^{m/2} \frac{2j-1}{2}, \tag{9}$$

revealing a clear pattern for coefficients associated with the highest moments of M_1*.*

DOI: 10.5540/03.2025.011.01.0483

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terms for 00-order coefficient
E[M1^02]:    0,50000000 * E[M1^( 2)]
terms for 02-order coefficient
E[M1^04]:   -0.25000000 * E[M1^( 4)] * E[M1^( 2)]
terms for 04-order coefficient
E[M1^04]:   -0.03125000 * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 2)]
E[M1^06]:    0.37500000 * E[M1^( 6)] * E[M1^( 2)] * E[M1^( 2)]
terms for 06-order coefficient
E[M1^04]:   -0.09375000 * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 2)]
E[M1^06]:   -0.28125000 * E[M1^( 6)] * E[M1^( 4)] * E[M1^( 2)] * E[M1^( 2)]
E[M1^08]:   -0.93750000 * E[M1^( 8)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)]
terms for 08-order coefficient
E[M1^04]:    2.87695313 * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 2)]
E[M1^06]:    1.82812500 * E[M1^( 6)] * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 2)] * E[M1^( 2)]
E[M1^06]:   -1.21875000 * E[M1^( 6)] * E[M1^( 6)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)]
E[M1^08]:    7.26562500 * E[M1^( 8)] * E[M1^( 4)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)]
E[M1^10]:    3.28125000 * E[M1^(10)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)]
terms for 10-order coefficient
E[M1^04]:   -1.76074219 * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 2)]
E[M1^06]: -126.50976563 * E[M1^( 6)] * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 2)] * E[M1^( 2)]
E[M1^06]:   -0.77343750 * E[M1^( 6)] * E[M1^( 6)] * E[M1^( 4)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)]
E[M1^08]:  -71.48437500 * E[M1^( 8)] * E[M1^( 4)] * E[M1^( 4)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)]
E[M1^08]:   -2.34375000 * E[M1^( 8)] * E[M1^( 6)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)]
E[M1^10]:  -88.59375000 * E[M1^(10)] * E[M1^( 4)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)]
E[M1^12]:  -14.76562500 * E[M1^(12)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)] * E[M1^( 2)]
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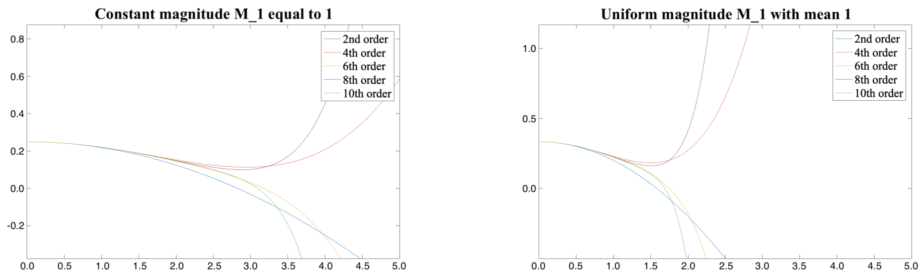
Figure 5: Computer output for terms for the Taylor expansion up to order 10. Source: Author.



(a) M_1 constant with mean 1 (b) M_1 uniform with mean 1

Figure 6: Graphs for Taylor approximations. Source: Author.

3.2 About the Convergence of the Taylor Expansion

We will continue to examine terms with the highest power of M_1 in each Taylor coefficient for the Lagrangian auto-covariance function, as they are most influenced by how heavy of the tail of M_1 is. These terms occur when $p_1 = p_2 = 1$ and $K_1^2 K_1^2 K_1^3 K_1^3 \dots K_1^{m/2} K_1^{m/2} K_1^{m/2+1} K_1^{m/2+1}$, and the total number of terms with these factors is $(m-1)(m-3)\cdots 3 \cdot 1$, for each even number $m \geq 2$.

Remark 3.11. *Note that taking the ratio between terms* $\mathbb{E}\left[\boldsymbol{\Theta}_1 \boldsymbol{\Theta}_1^T K_1^2 K_1^2 K_1^3 K_1^3 \dots K_1^{m/2+1} K_1^{m/2+1}\right]$ *in modulus, for decreasing and successive orders* $m+2$ *and* m *we get*

$$\frac{(m+1)(m-1)(m-3)\dots 3}{(m-1)(m-3)\dots 3} \frac{\mathbb{E}\left[M_1^{m+4}\right] \mathbb{E}\left[M_1^2\right]^{m/2+1}}{\mathbb{E}\left[M_1^{m+2}\right] \mathbb{E}\left[M_1^2\right]^{m/2}} \frac{2^{m/2+1}}{2^{m/2+2}} = \frac{m+1}{2} \frac{\mathbb{E}\left[M_1^{m+4}\right] \mathbb{E}\left[M_1^2\right]}{\mathbb{E}\left[M_1^{m+2}\right]}, \tag{10}$$

which matches with the pattern analyzed and established in the Remark (3.10).

Let us define a new Taylor series, formed by terms associated to the highest moments of the random amplitudes, which are associated with factors $\mathbb{E}\left[\boldsymbol{\Theta}_1 \boldsymbol{\Theta}_1^T K_1^2 K_1^2 K_1^3 K_1^3 \dots K_1^{m/2+1} K_1^{m/2+1}\right]$, for each $m \geq 2$, as in Eq.(5). Set

$$\Sigma'(t) = \sum_{\substack{m=2 \\ \text{m even}}}^{\infty} C_\Sigma^{\prime(m)} t^m = \sum_{\substack{m=2 \\ \text{m even}}}^{\infty} (-1)^{m/2} \frac{1}{(m/2)!} \frac{1}{2^{(3m+4)/2}} \mathbb{E}\left[M_1^{m+2}\right] \mathbb{E}\left[M_1^2\right]^{m/2} t^m. \tag{11}$$

Let $\rho > 0$ be the modulus of the ratio between decreasing and successive Taylor terms of $\Sigma' L(t)$. We can use Eq.(10) to obtain a simplified expression:

$$\rho = \left| \frac{C_{\Sigma}^{\prime(m+2)} t^{m+2}}{C_{\Sigma}^{\prime(m)} t^{m}} \right| = \frac{1}{4} \frac{1}{m+2} \frac{\mathbb{E}\left[M_1^{m+4}\right] \mathbb{E}\left[M_1^2\right]}{\mathbb{E}\left[M_1^{m+2}\right]} t^2. \tag{12}$$

Then, the Taylor series $\Sigma'(t)$ converges if $\rho < 1$ as $m \to \infty$. In other words, we have a condition on even moments of M_1 that guarantees the convergence of a Taylor series formed only by terms corresponding to the highest moments of M_1 in the expression of the Lagrangian auto-covariance.

Example 3.3. *Assuming that random wave number magnitude M_1 is deterministic and constant equal to 1, that is, $\mathbb{P}(M_1 = 1) = 1$. In this case, Eq.(12) leads us to*

$$\rho = \frac{1}{4} \frac{1}{m+2} \frac{1^{m+4} 1^2}{1^{m+2}} t^2 = \frac{1}{4} \frac{1}{m+2} t^2, \tag{13}$$

which goes to 0 as $m \to \infty$, for all $t > 0$. Therefore, the Taylor series $\Sigma'(t)$ converges for all $t \geq 0$.

4 Conclusion

In this work, we start with two-dimensional, mean-zero, homogeneous, stationary, and incompressible velocity fields expressed as a sum of finitely many Fourier modes. We establish conditions on the parameters of the model to obtain isotropic fields. We derive a few theoretical results about these random fields, particularly some concerning their Lagrangian auto-covariance function.

Assuming that the Lagrangian auto-correlation function can be represented as a Taylor series, we employ an appropriate programming language to obtain terms for small derivative orders and express them as symbolic expressions. We demonstrate the robustness of this methodology through several examples. Moreover, we identify some patterns, that allows us to analyze the convergence of a Taylor series formed by terms involving only the highest moments of the random magnitudes.

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