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Short Note on the Minimum Number of Distinct Eigenvalues of Unicyclic Graphs

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Abstract. Let $q(G)$ be the minimum number of distinct eigenvalues taken over all real symmetric matrices whose underlying graph is G. Using a linear time algorithm that diagonalizes any symmetric matrix associated to a unicyclic graph, we investigate the maximum eigenvalue multiplicity for these graphs. As an application, we determine the value of $q(G)$ for some family of tadpole graphs, which are unicyclic graphs formed by adding an edge between a cycle and a path.

Key words. Symmetric Matrix, Eigenvalue Multiplicity, Unicyclic Graph

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = \{1, 2, ..., n\}$ and edge set E. We can associate with G the collection of real $n \times n$ symmetric matrices defined by

$$S(G) = \{A : A = A^T \text{ and } a_{ij} \neq 0, \text{ for } i \neq j \Leftrightarrow \{i, j\} \in E\}. \tag{1}$$

We note that the diagonal entries of $A \in S(G)$ are free to be chosen as well as the nonzero values of a_{ij}, when $\{i, j\} \in E$. We denote by $q(A)$ the number of distinct eigenvalues of a square matrix A. For a given graph G, we define

$$q(G) = \min\{q(A) : A \in S(G)\}. \tag{2}$$

Clearly, if G has n vertices, then $1 \leq q(G) \leq n$. Moreover, $q(G) = 1$ if and only if G has no edges, since a matrix $A \in S(G)$ with exactly one eigenvalue is necessarily a scalar multiple of the identity matrix, which implies that G is the empty graph. The converse is clearly true. In the other case, $q(G) = n$ if and only if $G = P_n$, where P_n denotes the path on n vertices [8, Theorem 3.1]. In [1], it is shown that there is a great number of graphs G for which $q(G) = 2$. The value of $q(G)$ for some families of graphs, such as the join of a graph with itself, complete bipartite graphs and cycles, is also determined. Interestingly, there always exists a graph G on n vertices with $q(G) = k$, for each $k = 1, 2, \ldots, n$ [1, Corollary 3.6].

The knowledge of $q(G)$ for arbitrary graphs is related to a general problem in combinatorial matrix theory known as *the inverse eigenvalue problem for graphs*. Given a graph G, it consists of characterizing the possible spectra of A such that $A \in S(G)$ (see [10] for more details). It has received considerable attention by many researchers as it still remains an open problem in general.

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In this direction, it is well known that if $A(G)$ is the $(0,1)$ adjacency matrix of G and d is the diameter of G, then $q(A(G)) \geq d+1$. Furthermore, if G is a tree (a connected acyclic graph), then this result is true for all $A \in S(G)$, which means that $q(G) \geq d+1$ for trees [12]. While this inequality is tight for some trees, such as paths or stars, equality does not necessarily hold for any given tree. On the other hand, infinite families of trees for which equality holds are given in [2].

Our main interest lies in studying the value of $q(G)$ when G is a unicyclic graph, i.e, a connected graph with exactly one cycle. In this case, as we will see in this paper, it is possible to obtain $M \in S(G)$ such that $q(M) < d+1$ if the cycle length is even. Using a linear algorithm for symmetric matrices associated with unicyclic graphs, we determine $q(G)$ for a family of *tadpole graphs*, which are unicyclic graphs formed by adding an edge between a cycle and a path.

Our paper is structured as follows. The next section contains preliminary results concerning a known bound for $q(G)$. Section 3 presents the linear time algorithm which allows us to determine the eigenvalue multiplicity of unicyclic graphs in some cases as well as the value of $q(G)$ for tadpole graphs. The last section presents concluding remarks.

2 Preliminary Results

In [1], the authors presented a lower bound for $q(G)$ based on the length of an induced path of G. Recall that the length of a path is the number of edges in that path, and that the distance between two vertices (in the same component) is the length of a shortest path between those two vertices.

Theorem 2.1. *[1, Theorem 3.2] If there are vertices u, v in a connected graph G at distance d and the path of length d from u to v is unique, then*

$$q(G) \geq d+1. \tag{3}$$

From the above result, we can conclude that if G is a path on n vertices, then $q(G) = n$. Furthermore, if C_n denotes the cycle on n vertices, then $q(C_n) \geq \left\lceil \frac{n}{2} \right\rceil$. The exact value for $q(C_n)$ was also presented in [1].

Lemma 2.1. *[1, Lemma 2.7] Let C_n be the cycle on n vertices. Then*

$$q(C_n) = \left\lceil \frac{n}{2} \right\rceil. \tag{4}$$

This last result is derived by applying an algorithm presented in [7], based on the original algorithm from [6]. In [6], Ferguson presented an algorithm that builds a real symmetric $n \times n$ matrix A with given spectrum $\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \lambda_5 > \ldots$ such that $A \in S(C_n)$ and $a_{ij} > 0$ for all $\{i,j\} \in E$. In [7], the authors extend the work from [6] for Hermitian matrices whose graphs are cycles also considering the case where the spectrum is given by $\lambda_1 \geq \lambda_2 > \lambda_3 \geq \lambda_4 > \lambda_5 \geq \ldots$. In this case, all $a_{ij} > 0$ for all $\{i,j\} \in E$, except for the entry $a_{n-1,n}$, which is negative. The following result is an immediate consequence of this construction.

Corollary 2.1. *[7, Corollary 3.4] Any eigenvalue of a Hermitian matrix of a cycle has at most multiplicity 2.*

Let us denote the multiplicity of the eigenvalue θ of a symmetric matrix $A \in S(G)$ by $m_A(\theta)$.

Lemma 2.2. *[8, Corollary 2.3] Let P be a path that does not contain any edge of any cycle in G. Then*

$$m_{A(G\setminus P)}(\theta) \geq m_{A(G)}(\theta) - 1. \tag{5}$$

Lemma 2.2 implies Corollary 2.1 for symmetric matrices. In fact, $C_n \setminus \{v\}$ is a path, for any vertex $v \in V(C_n)$, and for any eigenvalue θ of $A \in S(C_n)$, we get $1 \geq m_{A(C_n \setminus \{v\})}(\theta) \geq m_{A(C_n)}(\theta) - 1$, and hence $m_{A(C_n)}(\theta) \leq 2$.

The same argument was used in [8] for other classes of graphs, such as the *tadpole graphs*. A (k, n)-tadpole graph is a graph on $n + k$ vertices formed by adding an edge between a vertex of cycle C_k and an endpoint of path P_n. Figure 1 illustrates a $(4, 3)$-tadpole graph.

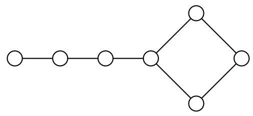

Figure 1: $(4, 3)$-tadpole graph. Source: From the authors.

By Lemma 2.2, we also have that any eigenvalue of a tadpole graph has multiplicity at most 2 [8, Lemma 5.1]. In the next section we investigate the value of $q(G)$ when G is a tadpole graph.

3 The Multiplicities of the Eigenvalues of a Unicyclic Graph

In [5], the authors presented a linear time algorithm named *DiagSMUnicyclic* (Algorithm 1) that computes in a given real interval the number of eigenvalues of a symmetric matrix $M \in S(G)$, where G is a unicyclic graph. Given $M \in S(G)$, we define $G(M)$ as the weighted graph associated to the matrix M. For a weighted graph $G(M)$ and for a fixed real number x, the algorithm computes the diagonal values of a diagonal matrix D *congruent* to $M + xI$. We recall that two square matrices A and B of order n are *congruent* if there is an invertible matrix P such that $A = P^T BP$. By Sylvester's Law of Inertia [11, Theorem 4.5.8], we have the following result.

Theorem 3.1. *[5, Theorem 2.2] If D is the diagonal matrix produced by the algorithm DiagSMUnicyclic$(G(M), -x)$, then the number of positive, negative and zero diagonal entries in D is equal to the number of eigenvalues of M which are greater, smaller and equal to x, respectively.*

For the execution of Algorithm 1, the cycle vertices $v_1, ..., v_k$ are ordered first, where each v_j is the root of the pendant tree T_j. Then, the vertices of T_j, for $j = 1, ..., k$ are ordered so that if v_c is a child of v_j, then $c > j$. Initially, $d_i := m_{ii} + x$ for all v_i. Then it processes the vertices of each T_j bottom-up, towards the root v_j (Step 2, Alg.1), calling algorithm *DiagonalizeW* (Alg.2). After the diagonalization of T_j if its root v_j has a child with zero value, then the edges of the cycle adjacent to v_j are removed if they haven't already been removed. Finally, the vertices of the cycle C_k, from v_k to v_1, are processed (Step 3 and 4, Alg. 1).

One of the advantages of this algorithm is that it allows us to determine the maximum multiplicity of the largest (or the smallest) eigenvalue for matrices of unicyclic graphs.

Proposição 3.1. *If G is a connected unicyclic graph, then the multiplicity of the largest (or the smallest) eigenvalue of a matrix $M \in S(G)$ is at most 2.*

Proof. Let x be the largest eigenvalue of $M \in S(G)$, where G is a unicyclic graph, and let us apply Algorithm *DiagSMUnicyclic*$(G(M), -x)$. By Theorem 3.1, all $d_j \leq 0$, since x is the largest eigenvalue of M. Assume that $d_j = 0$ for a vertex $v_j \neq v_1$, so that v_j has a parent v_k in G. Because d_j is 0, at the time v_k is processed, it has a child with value 0. If v_j is not a cycle vertex, then the algorithm assigns the positive value 2 to one child c of v_k, by Algorithm 2. Theorem 3.1 implies that M has an eigenvalue $\lambda > x$, a contradiction with the fact that x is the largest eigenvalue of

Algorithm 1: Algorithm *DiagSMUnicyclic*$(G(M), x)$ [5]

```
Input:  weighted unicyclic graph G(M) with ordered
        vertices v_1,...,v_n and scalar x
Output:  diagonal values d_1,...,d_n

 1.  Set d_i := m_ii + x, for all i
 2.  Apply DiagonalizeW(T_j, x) (Algorithm 2) to each tree T_j, 1 ≤ j ≤ k,
       skipping the initialization step already performed.
       If v_j has a child with zero value, remove the edges on the cycle
       adjacent to v_j, if they haven't already been removed.
 3.  If an edge v_{j-1}v_j, for some 1 ≤ j ≤ k, was removed in Step 2, then
       apply algorithm DiagonalizeW(P, x) to each path P that is not
       an isolated vertex choosing the endpoint of P with smaller
       index as the root.
 4.  If the cycle C_k was not disconnected in Step 2, then
       apply procedure DiagSMCycle(d_1, d_2, ..., d_k) (Algorithm 3).
```

Algorithm 2: *DiagonalizeW*$(T(M), x)$ [5]

```
Input:  weighted tree T(M) with ordered vertices
        v_1,...,v_n and scalar x
Output:  diagonal values d_1,...,d_n

Initialize d_i := m_ii + x, for all i
For j = n to 1
   if v_j is not a leaf then continue
       else if d_c ≠ 0, for all children c of v_j, then
           d_j ← d_j − Σ (m_cj)^2 / d_c   summing over all children of v_j
       else
           select one child c of v_j for which d_c = 0
           d_j ← −(m_cj)^2 / 2;  d_c ← 2;
           if v_j has a parent v_ℓ, remove the edge v_j v_ℓ.
end loop
```

M. If v_j is a cycle vertex and $v_j \neq v_2$, then by Algorithm 3 its diagonal value is replaced by 2, a contradiction again. In case that $v_j = v_2$, by the final step of Algorithm 3, the value of v_2 will be modified to 2 unless $z_2 = a_{12} = 0$ and $d_2 = d_1 = 0$, which means that x is an eigenvalue of M with multiplicity 2. Hence x is an eigenvalue of multiplicity 2 or all $d_j < 0$ except for the diagonal value d_1, which must be 0 at the end of the algorithm in the case that x is a simple eigenvalue. The proof for the smallest eigenvalue of M is analogous. □

We remark that for unicyclic graphs with an odd cycle it is known that at least one of the largest and smallest eigenvalues must be simple [3, Lemma 3.3].

The next result presents the value $q(G)$ for a (k, n)-tadpole graph G when $k < 2n$, obtained by applying algorithm *DiagSMUnicyclic*. We note that if k is odd, then any shortest path between two vertices of G is the unique shortest path. By Theorem 2.1 $q(G) \geq n + \frac{k-1}{2} + 1 = diam(G) + 1$.

Algorithm 3: Procedure $DiagSMCycle(d_1, d_2, ..., d_k)$ [5]

```
Set TriangDiagonalized := false; z_2 := m_12, z_k := m_1k,   z_j := 0, 3 ≤ j ≤ k−1.
For i = k to 3 do the following
    If d_i ≠ 0 then
        d_{i-1} ← d_{i-1} − (m_{i,i-1})^2/a_i; d_1 ← d_1 − (z_i)^2/d_i; z_{i-1} ← z_{i-1} − (m_{i,i-1}/d_i) z_i; //case 1
    else
        β ← 1/2 · (z_{i-1} − z_i · (d_{i-1}/(2 m_{i,i-1}) + m_{i,i-1})); γ ← z_i · (1 − d_{i-1}/(2(m_{i,i-1})^2)) + z_{i-1}/m_{i,i-1};
        d_i ← 2;   d_{i-1} ← −(m_{i,i-1})^2/2;   d_1 ← d_1 + 2β^2 · 1/(m_{i,i-1})^2 − γ^2/2;   //case 2
        if i ≥ 4 then
            z_{i-2} ← z_{i-2} − (m_{i-1,i-2}/m_{i,i-1}) · z_i;
            i ← i − 1;
        else
            TriangDiagonalized ← true;
end loop.
If TriangDiagonalized = false and z_2 ≠ 0 then
    if d_2 ≠ 0 then
        d_1 ← d_1 − (z_2)^2/d_2;
    else
        d_2 ← 2;   d_1 ← −(z_2)^2/2.
Return (d_1, d_2, ..., d_k).
```

On the other hand, if k is even, we can guarantee that G has two vertices having a unique shortest path whose length is $n + \frac{k}{2} - 1$. Thus, $q(G) \geq n + \frac{k}{2} = diam(G)$. Hence, $q(G) \geq n + \left\lceil \frac{k}{2} \right\rceil$. Next, we use the diagonalization algorithm to prove that in fact $q(G) = n + \left\lceil \frac{k}{2} \right\rceil$.

Theorem 3.2. *If G is a (k, n)-tadpole graph with $3 \leq k < 2n$ then*

$$q(G) = n + \left\lceil \frac{k}{2} \right\rceil. \tag{6}$$

Proof. If $k < 2n$, then $\left\lfloor \frac{k}{2} \right\rfloor \leq n - 1$. We note that independently of the matrix $M \in S(G)$, when we apply $DiagSMUnicyclic(G(M), -x)$ for x equals to one of the $n-1$ eigenvalues of path P_{n-1}, then the diagonal entry of the last vertex of P_{n-1}, counting from the pendant vertex of G, will be zero. Then, the algorithm will replace this zero value by 2 and it will replace the value of the last vertex of P_n by a negative one, removing the edge that connects the path P_n to the cycle C_k. Thus, using the algorithm from [7] we can obtain a principal submatrix M_1 of M with respect the cycle C_k such that $\left\lfloor \frac{k}{2} \right\rfloor$ eigenvalues of M_1 can be chosen among the $n-1$ eigenvalues of P_{n-1} and each one with multiplicity 2 in M_1. This implies that each of these $\left\lfloor \frac{k}{2} \right\rfloor$ eigenvalues of M_1 is an eigenvalue of M with multiplicity 2. Hence, if k is even, we have $q(M) \leq n + \frac{k}{2}$, and if k is odd we have that $q(M) \leq n + \frac{k-1}{2} + 1$. By Theorem 2.1 we conclude that $q(G) = n + \left\lceil \frac{k}{2} \right\rceil$. ☐

In order to prove the case where $k = 2n$, we need the following lemma. We write $M[G - v]$ to denote the submatrix of M obtained by deleting the row and the column corresponding to a vertex v of G.

Lemma 3.1. *Let C_k be a cycle on n vertices, $M \in \mathcal{S}(C_k)$ and $v \in C_k$. Suppose that λ is a simple eigenvalue of M and $M[C_k \setminus v]$. Then, there exist $N \in S(C_k)$ and $x \in \mathbb{R}$ such that $N_{ij} = M_{ij}$ if $(i,j) \neq (v,v)$, $N_{vv} = x$ and $m_N(\lambda) = 2$.*

Proof. Let u be a neighbor of v and view $M[C_k \setminus v]$ as rooted in u. Applying *DiagSMUnicyclic*$(M, -\lambda)$, $d_u^{(M)}$ is assigned with 0 because λ is an eigenvalue of $M[C_k \setminus v]$. Besides, $z_2^{(M)} = 0$ after the step $i = 3$ of Alg. 3. Indeed, if $z_2^{(M)} \neq 0$, $d_u^{(M)}$ would be replaced by 2 and $d_v^{(M)}$ by $-(z_2^{(M)})^2/2 \neq 0$, a contradiction, since λ is an eigenvalue of M by hypothesis. Let y be the final value assigned to $d_v^{(M)}$ at the end of the algorithm. We note that $y \neq 0$, since $m_M(\lambda) = 1$. Let $\delta \in \mathbb{R}$ and $N \in S(C_k)$ such that $N = M$, except for the entry N_{vv} which is equal to $M_{vv} + \delta$. Let $d_v^{(A)}$ denote the assignment of the vertex v in *DiagSMUnicyclic*$(A, -\lambda)$. Then, the algorithm *DiagSMUnicyclic*$(N, -\lambda)$ assigns $y + \delta$ to $d_v^{(N)}$ at the end of the algorithm. Indeed, in each step of *DiagSMUnicyclic*$(N, -\lambda)$ it will be true that $d_v^{(N)} = d_v^{(M)} + \delta$, $d_w^{(N)} = d_w^{(M)}$ for $w \in V(C_k) \setminus \{v\}$ and $z_\ell^{(N)} = z_\ell^{(M)}$ for $\ell \geq 2$. And, in the last step $d_v^{(N)} = d_v^{(M)} + \delta = y + \delta$. To conclude the proof, we only need to choose $x = M_{vv} - y$. □

We are now ready to prove the next result. We denote the spectrum of M (i.e., the multiset of its eigenvalues) by $Spec(M)$.

Theorem 3.3. *If G is a $(2n, n)$-tadpole graph, then*

$$q(G) = 2n. \tag{7}$$

Proof. Let $e = \{u, v\}$ be the edge connecting C_{2n} to P_n, where $u \in C_{2n}$ and $v \in P_n$. Let w be the neighbor of v in P_n and let $P_{n-1} = P_n \setminus v$. Finally, let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ be a list of any n real distinct numbers. By [9, Theorem 1] there exists $M_1 \in \mathcal{S}(P_{n-1})$ such that $Spec(M_1) = \{\lambda_1, \lambda_3, \lambda_4, \ldots, \lambda_n\}$ and $\{\lambda_2\} \subset Spec(M_1[P_{n-1} \setminus w])$. From the application of the algorithm from [6], there exists $M_2 \in \mathcal{S}(C_{2n})$ such that $Spec(M_2) = \{\lambda_1^{[2]}, \lambda_2, \lambda^*, \lambda_3^{[2]}, \lambda_4^{[2]}, \ldots, \lambda_n^{[2]}\}$ and $\{\lambda_2\} \subset Spec(M_2[C_{2n} \setminus u])$, where λ^* is a real number such that $\lambda_2 < \lambda^* < \lambda_3$.

Now, Lemma 3.1 guarantees there exists $x \in \mathbb{R}$, $x \neq (M_2)_{u,u}$ and $N \in S(C_{2n})$ such that $N = M_2$ for $(i,j) \neq (u,u)$, $N_{u,u} = x$ and $m_N(\lambda_2) = 2$. Let $M \in S(G)$ such that $M[C_k] = M_2$, $M[P_n \setminus v] = M_1$, $M_{u,v} = M_{v,u} = \delta > 0$ and $M_{v,v} = \lambda_2 + \frac{\delta^2}{x - M_{u,u}}$. Running *DiagSMUnicyclic*$(M, -\lambda_i)$ for $i \neq 2$. Since, λ_i is an eigenvalue of P_{n-1} then it will assign zero to d_w. Following that, we process the vertex v. Since w is a child of v with $d_w = 0$, then $d_v = -\frac{\delta^2}{2}$, $d_w = 2$ and the relation between v and its parent u is removed. Now *DiagSMUnicyclic*$(M, -\lambda_i)$ becomes the same as run *DiagSMUnicyclic*$(M[C_k], -\lambda_i)$. Since $m_{M_2}(\lambda_i) = 2$, then the algorithm will assign two zero values in the end of the execution, which implies that each λ_i for $i \neq 2$ is an eigenvalue of M with multiplicity 2. On the other hand, if we apply *DiagSMUnicyclic*$(M, -\lambda_2)$, since $\{\lambda_2\} \subset Spec([M_1[P_{n-1} \setminus w])$ then the algorithm assigns 0 to the child of w, and so it will replace this value by 2 and the value of d_w by a negative one, cutting the edge between w and v. Next, it processes the value of u by doing

$$d_u = M_{u,u} - \lambda_2 - \frac{\delta^2}{M_{v,v} - \lambda_2}, \tag{8}$$

which is equal to $x - \lambda_2$ by definition of $M_{v,v}$. From that point, *DiagSMUnicyclic*$(M, -\lambda_2)$ returns the same value as *DiagSMUnicyclic*$(N, -\lambda_2)$ when running the cycle, which will assign two zeros at end of the execution. Hence $m_M(\lambda_2) = 2$. Then $\{\lambda_1^{[2]}, \lambda_2^{[2]}, \ldots, \lambda_n^{[2]}\} \subset Spec(M)$. Since $|V(G)| = 3n$, there exists $\lambda_{n+1}, \ldots, \lambda_{2n}$, where $\lambda_i \neq \lambda_j$ for $i \in \{1, \ldots, n\}$, $j \in \{n+1, \ldots, 2n\}$ and such that $Spec(M) = \{\lambda_1^{[2]}, \lambda_2^{[2]}, \ldots, \lambda_n^{[2]}, \lambda_{n+1}, \ldots, \lambda_{2n}\}$. So, this implies that $q(M) \leq 2n$. By Theorem 2.1 we have that $q(G) = 2n$. □

4 Concluding Remarks

Using a linear time algorithm devised for symmetric matrices compatible with unicyclic graphs, we were able to determine the minimum number of distinct eigenvalues for a class of tadpole graphs. Even though it is true that $q(G) \leq n + \left\lceil \frac{k}{2} \right\rceil$ for a (k, n)-tadpole graph G [4, Corollary 49] we believe that the linear algorithm applied in this work may be an efficient tool for deriving new results concerning eigenvalue multiplicity of other classes of unicyclic graphs.

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