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Invariance Principle, S-procedure and the Asymptotic Behavior of T-S Fuzzy Systems

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Abstract. In this paper, the asymptotic behavior of nonlinear dynamical systems is studied by means of a T-S fuzzy system, which exactly represents the nonlinear system in question, and the extended Invariance Principle. We use the S-Procedure to obtain less conservative conditions than those presented in [2] to estimate the attracting invariant set in terms of Linear Matrix Inequalities.

Keywords. T-S Fuzzy Systems, Extended LaSalle Principle, Linear Matrix Inequality

1 Introduction

The Extension of LaSalle's Invariance Principle [1] provides information about the asymptotic behavior of trajectories of nonlinear dynamical systems under conditions less restrictive than Lyapunov's method. More specifically, it allows for the derivative of the candidate Lyapunov-like function to be positive in a bounded region. This extension was proved for several classes of dynamical systems, including the class of Takagi-Sugeno fuzzy systems, first for switched systems [6] and then for non-switched ones [2].

Exploring the fact that the T-S fuzzy modeling approach represents a large class of nonlinear systems by means of a sum of averaged linear models [4], the assumptions of the invariance principle were formulated in terms of linear matrix inequalities (LMIs), which are easily solved by convex programming techniques. The potential advantages of this approach are: (i) stability can be analyzed using LMIs in a Lyapunov formulation, and (ii) a fuzzy dependent Lyapunov function is usually less conservative than a quadratic one.

In this work, we build on top of [2], which allows the Lyapunov-like function $\dot{V}(x)$ to assume positive values in a bounded set, assuming that this set is contained inside the a bounded $l-$level set, and propose a method to obtain estimates of the attracting sets by means of LMI conditions. The proposed method is based on the application of the extended Invariance Principle along with the S-Procedure, which not only provides a set of LMI conditions with low levels of conservatism in comparison to [2], but ensure the existence of a bounded positive invariant set that attracts trajectories of the nonlinear system. The main advantage of the proposed result in comparison with those of [2] is that the LMI conditions obtained using the S-procedure ensure (as opposed to assuming) the existence of a l-level set that contains the bounded region where $V(x)$ is positive. The proposed result is useful for studying the asymptotic behavior of nonlinear systems modeled by T-S fuzzy system models that exactly represent the nonlinear system in question, estimating the attracting set and its region of attraction.

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2 Preliminaries

In this section, we introduce the main tools used to obtain conditions in terms of LMIs, which are less conservative than those presented in [2], to study the existence of attractors of nonlinear dynamical systems represented by Takagi-Sugeno fuzzy models.

2.1 T-S Fuzzy Systems

Lyapunov's results aid the stability analysis of nonlinear dynamical systems without the need to explicitly exhibit their solutions. However, finding functions satisfying the hypotheses of Lyapunov's theorems might be very difficult and often impossible. Even with less conservative results, such as those based on the Extension of the Invariance Principle [1], finding such functions is still a difficult task.

To overcome such difficulty, we explore T-S fuzzy systems, as they are capable of describing a class of nonlinear dynamical systems by means of a convex combination of local linear models, in a compact region of the state space. A consequence of this representation is to provide the assumptions of the results mentioned above by means of linear matrix inequalities, which are easily verified by numerical solvers, thus allowing the analysis of dynamics of non-linear systems in a simpler way.

In this work, we consider the class of nonlinear systems

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a complete $\mathcal{C}^1$ vector field, which can be exactly represented by a T-S fuzzy model in the set $Z = \{x \in \mathbb{R}^n : |x_\nu| \leq \bar{x}_\nu, \forall \nu \in \mathcal{N}\}$ as follows:

Model rule i:
If x_1 is M_{1i} and...and x_n is M_{ni}
Then $\dot{x}(t) = A_i x(t)$,

where $i = 1, ..., r$, M_{in} is the fuzzy set, r is the number of model rule, $x \in \mathbb{R}^n$ is the state vector and $A_i \in \mathbb{R}^{n \times n}$. The final output of the fuzzy system is inferred by using the center of gravity method for defuzzification

$$\dot{x}(t) = \sum_{i \in \mathcal{R}} h_i(x(t)) A_i x(t), \quad \mathcal{R} = \{1, ..., r\} \tag{2}$$

where

$$h_i(x(t)) = \frac{w_i(x(t))}{\sum_{i=1}^{r} w_i(x(t))} \tag{3}$$

$$w_i(x(t)) = \prod_{j=1}^{n} M_{ji}(x_j(t)) \tag{4}$$

$h_i(x(t))$ is regarded as the normalized weight of each model rule (called membership functions) and $M_{ji}(x_j(t))$ is the grade of membership of in $x_j(t)$ in M_{ji}, $\mathcal{R}$ is the set of indexes representing the r fuzzy rules, $\mathcal{N} = \{1, ..., n\}$ and $A_i \in \mathbb{R}^{n \times n}$ is a time invariant matrix. The topological boundary of Z is the set $\partial Z = \bigcup_{\nu \in \mathcal{N}} Z_\nu$, where $Z_\nu = \{x \in Z : x_\nu = \bar{x}_\nu\} \bigcup \{x \in Z : x_\nu = -\bar{x}_\nu\} \quad \forall \nu \in \mathcal{N}$. The membership functions $h_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\forall i \in \mathcal{R}$ are $\mathcal{C}^1$ and have the following convex properties:

$$\sum_{i \in \mathcal{R}} h_i(x) = 1, \quad h_i(x) \geq 0 \ \forall i \in \mathcal{R}, \ \forall x \in Z. \tag{5}$$

Our objective is to study the asymptotic behavior of (1) inside Z by studying the asymptotic behavior of (2) using the Extension of LaSalle's Invariance Principle [1]. The step-by-step guide to obtaining the T-S fuzzy model (2) from nonlinear system (1) can be seen in [5].

2.2 Candidate Lyapunov-type Function

We define a candidate auxiliary scalar $\mathcal{C}^1$ function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$, which depends on membership functions defined in the previous subsection:

$$V(x) = x' \sum_{k \in G} h_k(x) P_k x, \tag{6}$$

with $P_k = P_k' \in \mathbb{R}^{n \times n}, \quad \forall k \in G \subset \mathcal{R}$. Set G is a subset of $\mathcal{R}$, which can be conveniently chosen to include only a subset of the membership functions $h_k(x)$ in the candidate function. We will impose characteristics on matrices P_k with $k \in G$ such that we can use $V(x)$ to draw conclusions about the asymptotic behavior of (1). Taking the derivative of (6) yields:

$$\dot{V}(x) = \dot{x}' \sum_{k \in G} h_k P_k x + x' \sum_{k \in G} \dot{h}_k P_k x + x' \sum_{k \in G} h_k P_k \dot{x}. \tag{7}$$

Substituting (2) into (7) results in:

$$\dot{V}(x) = \underbrace{x' \sum_{j \in \mathcal{R}} \sum_{k \in G} h_k h_j (A_j' P_k + P_k A_j) x}_{(a)} + \underbrace{x' \sum_{k \in G} \dot{h}_k P_k x}_{(b)}. \tag{8}$$

The novelty of this work is to present sufficient conditions for both (a) and (b) to be negative outside a level set of V, thus ensuring that the derivative of V is negative outside this level set. Different from the results presented in [2], which required that (a) be negative in all $\mathbb{R}^n$. This improvement, which is obtained by exploiting the tool presented in Section 2.4, gives us the possibility of analyzing the solution for a larger class of systems.

2.3 Invariant Sets

Let us define the $L-$level set:

$$\Omega_L = \{x \in \mathbb{R}^n : V(x) < L\}. \tag{9}$$

In Subsection 2.2 in [2], it has been proven that if we choose $L < b$ with

$$b = \sum_{k \in G} \min_{x \in \partial Z} \lambda_{min}(P_k) h_k \min_{\nu \in \mathcal{N}} \bar{x}_\nu^2, \tag{10}$$

we guarantee that $\Omega_L \subset Z$, that is, we guarantee the limitation of Ω_L.

Now, let C be the set with positive derivative of the candidate function V:

$$C = \{x \in \Omega_L : \dot{V}(x) > 0\} \tag{11}$$

According to (8),

$$C = \{x \in \Omega_L : x' \big[\sum_{j \in \mathcal{R}} \sum_{k \in G} h_k h_j (A_j' P_k + P_k A_j) + \sum_{k \in G} \dot{h}_k P_k \big] x > 0\} \tag{12}$$

2.4 S-Procedure

The S-Procedure is a mathematical result that provides information about a specific inequality by analyzing another inequality. The earliest result of this kind is due to Finsler [3]. He proved that if A and B are two symmetric matrices and $x^T Bx = 0$ $(x \neq 0)$ implies $x^T Ax > 0$, then there exists a $\lambda \in \mathbb{R}$ such that $A + \lambda B$ is positive definite. Later, several applications and extensions of this result were presented, such as Yakubovich's result, which presented sufficient conditions to ensure the equivalence of the following two statements:
(i) $f(x) < 0$ and $g(x) \leq 0, \forall x \in E$, is not solvable;
(ii) there exists $\lambda > 0$ such that $f(x) + \lambda g(x) \geq 0, \forall x \in E$.

The result was first proved for quadratic functions [3] and later for other classes of functions, even for non-quadratic functions [7].In this work, we present sufficient conditions for (ii) to be satisfied, considering $f(x) = -x' \sum_{j\in\mathcal{R}} \sum_{k\in G} h_k h_j (A_j' P_k + P_k A_j)x$ and $g(x) = -(V(x) - l)$ with ℓ being a specific real number. In this way, ensuring that (i) is not solvable. This means that, the expression (a) of (8) will be negative definite outside the $l-$level set of V.

3 Conditions for the Existence of Invariant Sets

In this chapter, we present the main result, which provides sufficient conditions for the existence of an invariant set for the solution of a class of nonlinear systems. For this, we have to prove that the derivative of the function V given by (8) is definite negative outside a level set of the function V. Let's start by showing that expression (a) of (8) is definite negative outside this same level set, exploring the S-procedure. Consider the inequality

$$x' \sum_{k\in G} \sum_{j\in\mathcal{R}} h_k h_j \big(A_j' P_k + P_k A_j\big)x + \lambda(V(x) - l) \leq 0. \tag{13}$$

Now we add and subtract the following canceling terms containing slack variables $L_k, R_k \in \mathbb{R}^{n\times n}$

$$\begin{aligned} x' \sum_{k\in G} \sum_{j\in\mathcal{R}} \Big[& h_k h_j (A_j' P_k + P_k A_j) + h_k h_j (L_k A_j + A_j' L_k') \\ & - h_k h_j (A_j' L_k' + L_k A_j) + A_j' h_k h_j (R_k' + R_k) A_j \\ & - A_j' h_k h_j (R_k + R_k') A_j \Big] x + \lambda(V(x) - l) \leq 0, \end{aligned} \tag{14}$$

Substituting (6) into (14) we have

$$\begin{aligned} x' \sum_{k\in G} \sum_{j\in\mathcal{R}} \Big[& h_k h_j (A_j' P_k + P_k A_j) + h_k h_j (L_k A_j + A_j' L_k') \\ & - h_k h_j (A_j' L_k' + L_k A_j) + A_j' h_k h_j (R_k' + R_k) A_j \\ & - A_j' h_k h_j (R_k + R_k') A_j \Big] x + \lambda \sum_{k\in G} x' h_k P_k x - \lambda l \leq 0, \end{aligned} \tag{15}$$

From (5) we can write

$$x'\sum_{k\in G}\sum_{j\in\mathcal{R}}\Big[h_kh_j(A_j'P_k+P_kA_j)+h_kh_j(L_kA_j+A_j'L_k') - h_kh_j(A_j'L_k'+L_kA_j)+A_j'h_kh_j(R_k'+R_k)A_j - A_j'h_kh_j(R_k+R_k')A_j\Big]x+\lambda\sum_{j\in\mathcal{R}}h_j\sum_{k\in G}x'h_kP_kx-\lambda l\leq 0, \quad (16)$$

Rearranging (16) into matrix form, we finally obtain

$$\sum_{j\in\mathcal{R}}\sum_{k\in G}h_kh_j\begin{bmatrix}x' & x'A_j' & 1\end{bmatrix}\Upsilon_{kj}\begin{bmatrix}x\\ A_jx\\ 1\end{bmatrix}\leq 0 \quad (17)$$

where

$$\Upsilon_{kj}=\begin{bmatrix}L_kA_j+A_j'L_k'+\lambda P_k & * & 0\\ P_k-L_k'+R_kA_j & -R_k-R_k' & 0\\ 0 & 0 & -\lambda l\end{bmatrix}. \quad (18)$$

Therefore, we can conclude that (13) is satisfied if there exist real numbers $\lambda, l > 0$ and matrices P_k, L_k, R_k satisfying (18) for all $k \in G$ and $j \in \mathcal{R}$. In the next subsection, we present conditions so that expression (b) of (8) is also negative outside the l-level set of V.

3.1 Reducing the Conservativeness of LMIs

In order to satisfy (17), it suffices that

$$\Upsilon_{kj}<0 \quad \forall k\in G, j\in\mathcal{R} \quad (19)$$

But we can make this LMI less conservative by transforming it into

$$\Upsilon_{kk}<0 \qquad \forall k\in G \quad (20)$$

$$\Upsilon_{kj}+\Upsilon_{jk}<0 \quad \forall k,j\in G, k\neq j \quad (21)$$

$$\Upsilon_{kj}<0 \quad \forall k\in G, j\in\mathcal{R}-G \quad (22)$$

In order to study the possibilities for a positive derivative of V, we define the following sets:

$$\mathcal{D}=\{x\in\Omega_L : x'\sum_{k\in G}\dot{h}_kP_kx>0\}, \quad (23)$$

$$\mathcal{P}=\{x\in\Omega_L : x'\sum_{j\in\mathcal{R}}\sum_{k\in G}h_kh_j(A_j'P_k+P_kA_j)x>0\}. \quad (24)$$

We can readily infer that $\mathcal{P}\cap\mathcal{D}\subset C$. Also, the following statements are true: $\mathcal{P}^C\cap\mathcal{D}\supset C$, $\mathcal{P}\cap\mathcal{D}^C\supset C$ and $\mathcal{P}^C\cap\mathcal{D}^C\subset C^C$. From that we conclude that $C\subset(\mathcal{P}\cup\mathcal{D})$

From the LMIs, we guarantee that expression (a) of (8) is negative outside the l-level set of V, that is, $\mathcal{P}\subset\Omega_l$. If we choose l satisfying

$$\sup_{x\in\mathcal{D}}V(x)<l<L, \quad (25)$$

we force the expression (b) of (8) also be negative outside the l-level set of V, that is, $C\subset\Omega_l$ and since $\Omega_l\subset\Omega_L$, we have $C\subset\Omega_l\subset\Omega_L$.

Theorem 3.1. *Consider the nonlinear system* (1)*, which can be exactly represented by the T-S Fuzzy system* (2) *inside the bounded set* Z*. Let* L *such that* Ω_L *is bounded and* $\Omega_L \subset Z$*. If there exist* P_k, L_k, R_k *and* $l < L$ *such that* (20)*,* (21)*,* (22) *and* (25) *are satisfied, then every solution of* (1) *starting in* Ω_L*, converges to the largest invariant set of* (1) *inside* $E \coloneqq \{x \in \Omega_L : \dot{V}(x) = 0\} \bigcup \Omega_l$*.*

Proof. Considering Z and $V(x)$ as in (6), we can choose $L < b$ as defined in (10) to guarantee that $\Omega_L \subset Z$, that is, Ω_L is bounded. Assuming the existence of $l < L$ and matrices P_k, L_k, R_k satisfying (20), (21), (22) and (25) with $\mathcal{D}$ as in (23), we have that the expressions (a) and (b) of (8) are negative outside of l-level set of V. Thus, as described in Section 3.1, the set C will be a subset of Ω_l. Therefore, the result follows from Theorem 2 from [1]. □

Example 3.1. *Consider the following nonlinear system:*

$$\dot{x} = \begin{bmatrix} -10x_1 + \frac{30}{25}(x_1^2 + x_2^2 - 25)x_2 \\ -20x_2 \end{bmatrix}. \tag{26}$$

Using the sector nonlinearity approach [4], we obtain a T-S Fuzzy Model that exactly represents (26) *in the set* $Z = \{x \in \mathbb{R}^n : x_1 \in [-5, 5],\ x_2 \in [-5, 5]\}$*:*

$$\dot{x} = \sum_{i \in \mathcal{R}} h_i A_i x, \quad \mathcal{R} = \{1, 2\} \tag{27}$$

$$A_1 = \begin{bmatrix} -10 & 30 \\ 0 & -20 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -10 & -30 \\ 0 & -20 \end{bmatrix} \tag{28}$$

$$h_1(x) = \frac{x_1^2 + x_2^2}{50}, \quad h_2(x) = 1 - \frac{x_1^2 + x_2^2}{50} \tag{29}$$

Solving the LMIs defined in (20)*,* (21) *and* (22) *using convex programming techniques, the following results were obtained*

$$P_1 = \begin{bmatrix} 0.0284 & -0.0000 \\ -0.0000 & 10.0250 \end{bmatrix} \qquad L_1 = \begin{bmatrix} 0.0201 & -0.0000 \\ 0.0000 & 0.0645 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 0.0029 & -0.0000 \\ -0.0000 & 0.4980 \end{bmatrix} \tag{30}$$

with $G = \{1\}$*,* $b = 0.35$ *and* $l = 0.175$*. The feasibility of the LMI conditions guarantees that the solutions of the nonlinear system* (26) *starting inside* Ω_L*, with* $L < 0.35$ *converge to the largest invariant set inside* $E \coloneqq \{x \in \Omega_L : \dot{V}(x) = 0\} \bigcup \Omega_l$*. Figure 1 illustrates the relevant sets. The outer red line represents* Ω_L*, the inner red line represents* Ω_l *and the blue dots represent set* $\mathcal{D}$ *as defined in* (23)*.*

4 Final Considerations

In this work, we explored the S-procedure and the extended Invariance Principle for analyzing the asymptotic behavior of nonlinear systems represented by a T-S fuzzy model. As in [2], we allowed $\dot{V}(x)$ to assume positive values in a bounded set and proposed methods to obtain estimates of the invariant sets using LMIs. However, the use of the S-procedure, along with rewriting the obtained LMIs, has drastically reduced the conservativeness of the existing results. In future works, we aim to further explore the characteristics of T-S Fuzzy systems, such as membership functions and their derivatives, aiming to express all main assumptions of the invariance principle directly in the LMIs.

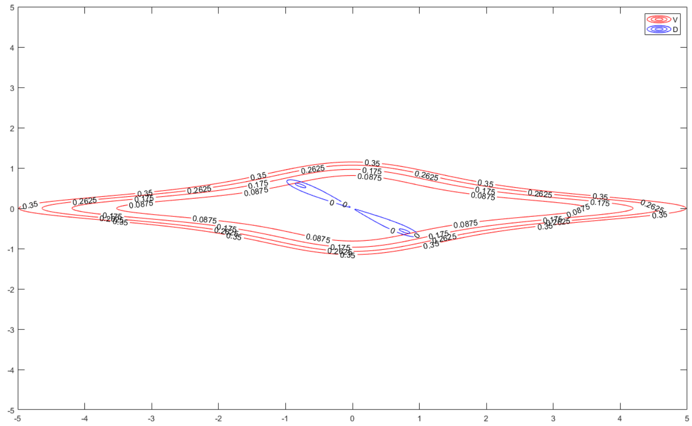

Figure 1: Sets Ω_L, Ω_l and $\mathcal{D}$. Source: Authors.

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