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Computational Simulations of the Buckley-Leverett Equation Applied to the Flow of Two-Phase Fluids

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Abstract. The purpose of this work was to investigate the flow of two-phase fluids via the Buckley-Leverett equation, including dispersive and diffusive terms. For this, a weighted essentially non-oscillatory scheme, a Runge-Kutta method and a finite difference scheme were computationally implemented. The use of these methods made it possible to obtain numerical solutions, without excessive numerical dispersion and dissipation, sufficient to assist in the understanding of the mixing profiles of saturated water and petroleum fluids, inside pipelines filled with porous material. In addition, the impact of adding such terms in the original equation has been studied.

Keywords. Partial Differential Equations, Numerical Methods, Fluid Dynamics

1 Introduction

The technological advances achieved after the Industrial Revolution, followed by the development of equipment dependent on energy sources from petroleum, made the oil industry reach a crucial role in the world economy. As it is an exhaustible natural source and a potential pollutant, if disposed of in an erroneous way, the efficiency in the extraction process and the detailed understanding of the phenomena involved become indispensable for the challenges of delivering a product inserted within a clean economy.

A central problem in this area is the displacement of petroleum through pipes filled with a porous medium, characterized by the injection of another fluid (saturated water) to help maintain the flow inside the tube. In this sense, the mathematical model used to describe the flow of two-phase incompressible fluids is the classical Buckley-Leverett equation [1]. The property of nonlinearity of the partial differential equation enables the use of numerical methods and computational techniques, in an attempt to find approximate solutions without spurious oscillations (numerical dispersion) or excessive numerical dissipation (numerical diffusion).

In preliminary studies, the authors of this work investigated a fifth-order weighted essentially non-oscillatory scheme applied in the classical Buckley-Leverett equation. Furthermore, the addition of a diffusive term was performed [2].

In this way, the objective of this work was to include a dispersive term to carry out a study on the influence of the dispersive and diffusive terms on the numerical solutions of the classical Buckley-Leverett equation. For that, a weighted essentially non-oscillatory scheme, coupled to a three-stage Runge-Kutta method and a central finite difference scheme were implemented in the discretization of the modified Buckley-Leverett equation. Finally, numerical solutions capable of representing the temporal evolution of the phenomenon were calculated. Own codes were made

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in Octave and with its, the results are achieved regarding the evolution of the mixture between saturated water and oil.

2 Mathematical Modeling

Currently, a problem of global interest is the extraction of petroleum underground through a tube filled with a porous medium. After drilling the soil to the underground oil reservoir, a certain amount is drained due to the high pressure that the oil is found, but as the extraction progresses, there is a decrease in pressure with consequent interruption of the flow, still leaving a lot of petroleum in the subsoil. A standard method subsequent to the initial extraction is to pump water into rest oil reservoir to force the continuation of extraction. In this case, the fluid is two-phase, oil and water, and the flow is restarted in the porous medium consisting of rock or sand.

The mathematical modeling that will be described consists of representing the oil one-dimensional flow through the tube filled with porous material, by water pumping [4]. Such a model was initially proposed by [1], in studies on the flow of two-phase incompressible fluids in porous media.

Let $0 \leq q(x,t) \leq 1$ be the fraction of saturated water and $1-q(x,t)$ the fraction of oil contained in a pipe filled with a porous material. Such fluids are essentially incompressible, which ensures that the total flow between the pipe ends is equal to any smaller portion of the pipe.

In this way, in regions of the tube where $q = 0$ (pure oil) or $q = 1$ (pure water), the velocities are constant and distinct, but when $0 < q < 1$, the difference between the surface tensions of fluids causes them to move and mix. A model in which the rate of change of q over time (q_t) is described by the following conservation law,

$$q_t + f(q)_x = 0, \tag{1}$$

in which $f(q) = \dfrac{q^2}{q^2 + a(1-q^2)}$ is the water flux, $0 < a < 1$ represents the porosity of the medium and $1 - f(q)$ is the oil flux, with $q = q(x,t)$. Equation (1) models a flow from left to right, in which the tube thickness does not influence the dynamics in question.

The reestablishment of the oil flow, from left to right, can be done by filling part of the pipe on the left with saturated water, allowing the resumption of oil extraction. An initial condition for modeling that procedure is

$$q(x,0) = \begin{cases} 1, & if\ x < \delta \\ 0, & if\ x > \delta \end{cases} . \tag{2}$$

The initial condition, equation (2), can be approximated by the following function:

$$q(x,0) = 1 - \left[\frac{1 + tanh\left(\alpha(x-\delta)\right)}{2}\right], \tag{3}$$

in which δ is a parameter associated with the position of the function and α corresponds to how quickly the function varies from zero to one.

Let be the spatial domain $[x_i, x_f]$, the boundary condition on the left was of the *Dirichlet* type and on the right of the radiation type, that is, when the dynamics reach the boundary, the flow simply goes through the boundary, is not being affected by the edge.

Considering the effects of infiltration in porous media, the following modification to equation (1) is

$$q_t + f(q)_x = \varepsilon q_{xx} + \varepsilon^2 \kappa q_{xxt}, \tag{4}$$

where ε is a diffusibility coefficient and κ is a dispersive coefficient.

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From the point of view of numerical methods, the equation (4) is rewritten as follows

$$\left(q - \varepsilon^2 \kappa q_{xx}\right)_t + f(q)_x = \varepsilon q_{xx}, \tag{5}$$

and then

$$\begin{cases} p_t + f(q)_x = \varepsilon q_{xx} \\ p = \left(q - \varepsilon^2 \kappa q_{xx}\right) \end{cases}. \tag{6}$$

The choice of numerical methods to solve equation (6) is essential, as a numerical scheme that has a high degree of numerical dissipation can stand out and mask the real effect of the dispersive and diffusive terms of equation (4), impairing the interpretation of the results.

3 Numerical Methods

For this work, a weighted essentially non-oscillatory scheme (WENO-5 method), a central finite difference scheme (CFDS-4) were chosen for spatial discretization and the third-order Runge-Kutta TVD method (Total Variation Diminishing) was chosen for temporal discretization.

3.1 Non-oscillatory Schemes and WENO-5

In general, a rth-order essentially non-oscillatory (ENO) scheme chooses the smoothest stencil among r possibilities and uses only the chosen one to approximate the flow [3]. The idea of weighted essentially non-oscillatory (WENO) schemes is to find a convex combination of all candidate stencils, for the numerical flow approximation. A weight is assigned to each stencil, representing its contribution to the process.

The WENO schemes technique is based on the flow version of the ENO schemes, considering a one-dimensional conservation law $u_t + f(u)_x = 0$. The spatial operator that approximates $-f(u)_x$ in x_j is given by

$$L \quad = \quad -\frac{1}{\Delta x}(\hat{f}_{j+1/2} - \hat{f}_{j-1/2}), \tag{7}$$

in which Δx is the size of the spatial discretization and $\hat{f}_l$ is the numerical flux.

ENO schemes approximate $\hat{f}_{j+1/2}$ through a polynomial interpolation at the points of each stencil [3]. This approximation is given by

$$\hat{f}_{j+1/2} = q_k^r(f_{j+k-r+1}, \ldots, f_{j+k}), \text{ with, } q_k^r(g_0, \ldots, g_{r-1}) = \sum_{l=0}^{r-1} a_{k,l}^r g_l. \tag{8}$$

Let $g(x)$ be a smooth function. The average approximation of $g(x)$ in cell I_j is defined by

$$\bar{g}_j = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} g(\xi) d\xi \quad \text{where} \quad \xi \in I_j = \left(x_{j-1/2}, x_{j+1/2}\right). \tag{9}$$

To obtain the constants $a_{k,l}^r$ in (8), consider the primitive function of $g(x)$ defined by $G(x) = \int_{-\infty}^{x} g(\xi) d\xi$. The value of $G\left(x_{j+1/2}\right)$ can be written as

$$G\left(x_{j+1/2}\right) = \sum_{i=-\infty}^{j} \int_{x_{i-1/2}}^{x_{i+1/2}} g(\xi) d\xi = \sum_{i=-\infty}^{j} \bar{g}_i \Delta x_i. \tag{10}$$

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Equation (10) means that, once the average approximations of the cells $\bar{g}_i$ are known, the values of $G(x)$ at the boundary of cell I_i are also known. So, the constants $a_{k,l}^r$ are determined by interpolating $G\left(x_{j+1/2}\right)$ by a r degree polynomial $P(x)$, at most. Therefore,

$$\hat{f}_{j+1/2} = p\left(x_{j+1/2}\right) = P'\left(x_{j+1/2}\right) = \sum_{l=0}^{r-1} a_{k,l}^r g_l, \tag{11}$$

where $a_{k,l}^r$ are obtained from the Lagrange interpolating polynomial [3], with the data in Table 1.

Table 1: $a_{k,l}^r$ coefficients.

r	k	$l=0$	$l=1$	$l=2$
3	0	1/3	-7/6	11/6
	1	-1/6	5/6	1/3
	2	1/3	5/6	-1/6

A rth-order ENO scheme implies to a $(2r-1)$th-order WENO scheme, so a 3rd-order ENO scheme leads to a 5th-order WENO scheme. In WENO schemes, for each candidate stencil S_k, $k = 0, 1, \ldots, r-1$, a weight ω_k is assigned and these are used to calculate the numerical flux

$$\hat{f}_{j+1/2} = \sum_{k=0}^{r-1} \omega_k q_k^r(f_{j+k-r+1}, \ldots, f_{j+k}). \tag{12}$$

The weight ω_k for the stencil S_k is defined by

$$\omega_k = \frac{\alpha_k}{\alpha_0 + \ldots + \alpha_{r-1}}, \text{ with } \alpha_k = \frac{C_k^r}{(\varepsilon + IS_k)^p}, \quad k = 0, 1, \ldots, r-1. \tag{13}$$

Taking $p = r$, the coefficients C_k^r are optimal values to determine ω_k. The term IS_k is an indicator of smoothness and for $r = 3$ we have

$$\begin{array}{rcl} IS_0 & = & \frac{13}{12}(f_{j-2} - 2f_{j-1} + f_j)^2 + \frac{1}{4}(f_{j-2} - 4f_{j-1} + 3f_j)^2 \\ IS_1 & = & \frac{13}{12}(f_{j-1} - 2f_j + f_{j+1})^2 + \frac{1}{4}(f_{j-1} - f_{j+1})^2 \\ IS_2 & = & \frac{13}{12}(f_j - 2f_{j+1} + f_{j+2})^2 + \frac{1}{4}(3f_j - 4f_{j+1} + f_{j+2})^2 \end{array}. \tag{14}$$

This measure was introduced by [3], with the aim of achieving high accurate for the case where $r = 3$. Note that as IS_k increases, the smoothness decreases and, consequently, α_k becomes close to zero as does ω_k, meaning that a weight close to of zero will be assigned to non-smooth solutions.

3.2 Third-order Runge-Kutta TVD

Once the spatial discretization is concluded, a method for temporal discretization that maintains the non-oscillatory characteristics achieved is necessary.

Numerical methods belonging to the TVD class (Total Variation Diminishing) have the property of avoiding oscillations that are not typical of the phenomenon under study [4]. A good alternative is the high-order Runge-Kutta TVD methods, which were developed by [5] in research related to efficient implementations for ENO's schemes.

A method is called Total Variation Diminishing (TVD) if, for any data set U^n, the values U^{n+1} computed by the method satisfy $TV(U^{n+1}) \leq TV(U^n)$, where

$$TV(U^n) = \sum_{i=1}^{N} |U_i^n - U_{i-1}^n| \tag{15}$$

is the total variation. In this work, the third-order Runge-Kutta TVD (RK3-TVD) method was chosen, whose expressions are given by

$$\begin{array}{lcl} u^{(1)} & = & u^n + \Delta t L(u^n) \\ u^{(2)} & = & \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}) \\ u^{n+1} & = & \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}), \end{array} \tag{16}$$

in which L is the spatial operador of differential equation.

The WENO-5 and RK3-TVD methods numerically solve the classical Buckley-Leverett equation. To discretize the diffusive and dispersive terms of the modified Buckley-Leverett equation, we use the central finite difference scheme.

3.3 Fourth-order Central Finite Difference Scheme

There are at least two ways to numerically add the U term in the conservation law: one is to discretize the diffusive term in the flux context [2],

$$q_{xx}(x_{i+1/2}, t) \cong \frac{\bar{q}_{i+3/2} - 2\bar{q}_{i+1/2} + \bar{q}_{i-1/2}}{\Delta x^2}, \tag{17}$$

and the other in the finite difference context [6],

$$q_{xx}(x_i, t) \cong \frac{-\bar{q}_{i-2} + 16\bar{q}_{i-1} - 30\bar{q}_i + 16\bar{q}_{i+1} - \bar{q}_{i+2}}{12(\Delta x^2)}, \tag{18}$$

In this work, both discretizations were performed, as for the interpretation of the results there were no significant differences between them, we chose to keep the fourth-order central finite difference scheme (CFDS-4) in our codes.

This method was also used to discretize the second equation in equation (6), differently of [6] that use a staggered mesh and numerical integration to incorporate such equation into WENO scheme. In this way, the dispersive term enters the temporal evolution by replacing the term u^n of RK3-TVD method by $v^n = u^n - \varepsilon^2\kappa q_{xx}(x_i, t^n)$, in which $q_{xx}(x_i, t^n)$ is defined by equation (18). Details about stability those methods can be founded in a study of stability analysis presented in the reference [2].

4 Simulations and Results

The simulations were performed by initial conditions defined in Section 2. For all scenarios, we have $x \in [-1, 1]$ with 256 subintervals, $\Delta x = 1/124$, and $\Delta t = 0.1\Delta x$. Furthermore, $a = 0.5$ was assigned to the constant that characterizes the porous medium, equation (1).

The initial condition is given by $q(x, 0) = 1 - \left[\frac{1 + tanh\,(50(x + 0.5))}{2}\right]$, Figure 1, magenta color (label IC). In all graphs we have a fluids mix when q assumes values between $q = 1$ (pure water) and $q = 0$ (pure petroleum). The comparison of the graphs reveals the emergence of a region where the fluids mix, between the values of $q = 1$ (pure water) and $q = 0$ (pure petroleum), while the fluid dynamics develops to on the right.

The numerical solution obtained after 400 iterations is showed in Figure 1. In magenta color, we have the initial condition (IC); in black color we have the solution obtained for the classical Buckley-Leverett equation (BL), that is, $\varepsilon = \kappa = 0$; in blue color we have the solution for the Buckley-Leverett equation with diffusive term (MBL), that is, $\varepsilon = 0.01$ and $\kappa = 0$, and in red color

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we have the solution for the complete Buckley-Leverret equation (TMBL), that is, $\varepsilon = 0.01$ and $\kappa = 8.0$.

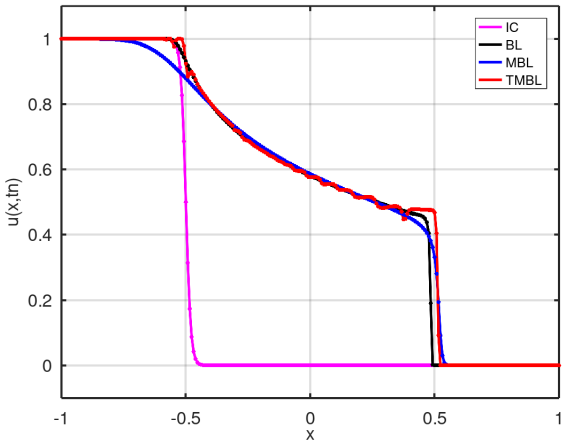

Figure 1: Initial condition and numerical solution after 400 iterations. Source: The authors.

To observe the influence of added terms two zooms were done in $x \cong -0.5$, Figure (2) left and right.It is noted that the effect of the diffusive term was to spread the mixture between oil and water more, when comparing the BL solution (black color) and the MBL solution (blue color), while the dispersive term was to add oscillations in the mixture between oil and water, TMBL solution (red color) and BL solution (black color).

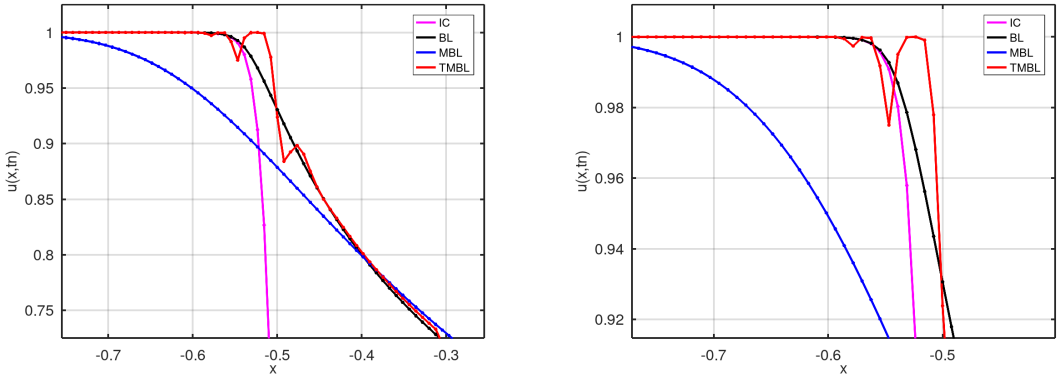

Figure 2: Zoom 1. Source: The authors.

The same can be concluded in the region of $x \cong 0.3$, with a second zoom, Figure (3). The temporal evolutions follow a similar propagation speed, they all have a more accentuated transition close to $x \cong 0.5$, however, there is greater spread in the mixture in the solution that only includes the diffusive term (MBL solution, blue color), compared to the others, and there is an oscillatory dispersive behavior in the complete solution (TBML, red color).

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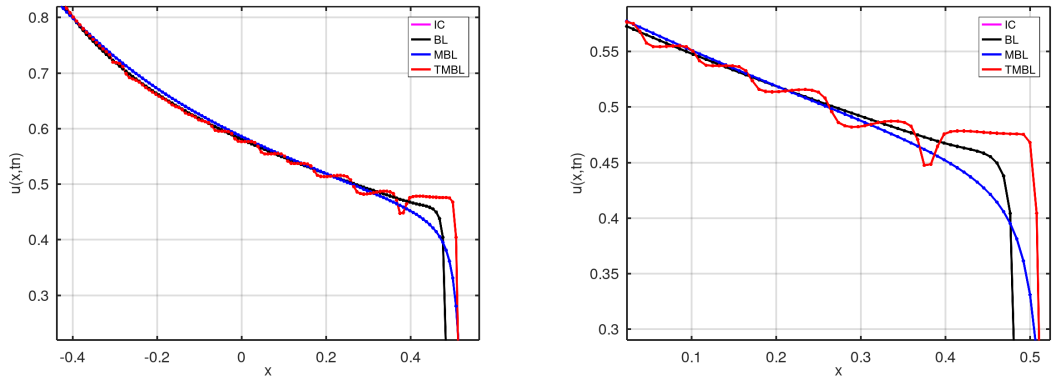

Figure 3: Zoom 2. Source: The authors.

5 Final Considerations

In this study, the WENO-5, CFDS-4 and RK3-TVD methods were applied to the Buckley-Leverett Equation with dispersive and diffusive terms, in order to investigate the temporal evolution of two-phase fluid flow. The motivation arises from the fact that during petroleum extraction it is necessary to inject a second fluid (saturated water) in order to maintain the pressure in the pipe.

The mathematical model considered, together with the numerical methods implemented, proved to be suitable to provide approached solutions sufficiently close to the analytical solution. Simulations were performed and the results showed that the inclusion of the diffusive and dispersive terms makes it possible to model different mixture profiles, displaying the different dynamics that occur in regions where there are transitions between water and oil.

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