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A Non-Uniform Herd Approach to Bilevel Optimization

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Abstract. In the present paper we discuss bilevel optimization. We consider a recent ϵ-approximation to the pessimistic bilevel optimization and we show that it may actually converge to the solution of the optimistic bilevel optimization problem. We also propose an ϵ-approximate smoothed problem which may model more realistic situations.

Keywords. Bilevel Optimization, Pessimistic Bilevel Optimization, Optimization

1 Introduction

Bilevel optimization [1] problems model decentralized games where the actions of a leader trigger the actions of a follower, both the follower and the leader try to minimize their costs (or maximize gains), and each of them has his/her definition of what kind of cost to minimize. The leader's actions constrain the follower's actions and influence how the follower computes his costs. The follower's decisions, in turn, affect how the leader calculates her costs but do not constrain the possibilities of the leader.

One can write the optimistic formulation of a bilevel problem as follows:

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{y}} \quad & f(\boldsymbol{x},\boldsymbol{y}) \\ \text{s.t.:} \quad & \boldsymbol{x} \in X \\ & \boldsymbol{y} \in \mathcal{F}(\boldsymbol{x}), \end{aligned} \tag{1}$$

where $\mathcal{F}(\boldsymbol{x})$ is the (follower's) **rational response set**, defined as follows:

$$\mathcal{F}(\boldsymbol{x}) := \arg\min_{\boldsymbol{y}\in Y(\boldsymbol{x})} g(\boldsymbol{x},\boldsymbol{y}). \tag{2}$$

It will be helpful later to define the **rational response cost** as

$$g(\boldsymbol{x}) := \min_{\boldsymbol{y}\in Y(\boldsymbol{x})} g(\boldsymbol{x},\boldsymbol{y}). \tag{3}$$

That is, one can rewrite the rational response set as

$$\mathcal{F}(\boldsymbol{x}) = \{\boldsymbol{y} \in Y(\boldsymbol{x}) : g(\boldsymbol{x},\boldsymbol{y}) = g(\boldsymbol{x})\}. \tag{4}$$

We call the above formulation optimistic because the leader can select the follower's rational response that gives the best possible result for his purposes. Another interpretation is that one assumes that the follower has information about the leader's objective and is willing to collaborate in minimizing the leader's objective function, as long as this collaboration does not cause extra losses to the follower.

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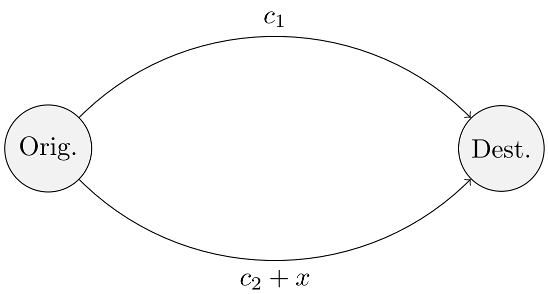

Figure 1: Illustration of a simple toll-setting problem. There is one destination, one origin, and two paths. The untolled way has cost c_1. The tolled path has cost c_2 plus the toll fee x. Source: produced by the author.

Assuming that the follower will select the most favorable rational response to the leader might not model reality well enough. It might be the case that the follower is willing to antagonize the leader. A proposed alternative is the pessimistic formulation of the bilevel optimization problem. To introduce the pessimistic bilevel [3] problem, let us define the **maximal response cost**:

$$\bar{f}(\boldsymbol{x}) := \max_{\boldsymbol{y} \in \mathcal{F}(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{y}). \tag{5}$$

Then, we define the **pessimistic response set** as:

$$\mathcal{P}(\boldsymbol{x}) := \big\{\boldsymbol{y} \in \mathcal{F}(\boldsymbol{x}) : f(\boldsymbol{x}, \boldsymbol{y}) = \bar{f}(\boldsymbol{x})\big\}. \tag{6}$$

Finally, the pessimistic bilevel problem is given by

$$\begin{aligned} \min_{\boldsymbol{x}, \boldsymbol{y}} \quad & f(\boldsymbol{x}, \boldsymbol{y}) \\ \text{s.t.:} \quad & \boldsymbol{x} \in X \\ & \boldsymbol{y} \in \mathcal{P}(\boldsymbol{x}). \end{aligned} \tag{7}$$

Let us, for example, consider a simple toll-setting problem of the form

$$\begin{aligned} \min_{x, y_1, y_2} \quad & -xy_2 \\ \text{s.t.:} \quad & x \geq 0 \\ & \boldsymbol{y} \in \arg\min_{\substack{y_1 + y_2 = 1 \\ y_1,\ y_2 \geq 0}} \quad c_1 y_1 + (c_2 + x) y_2. \end{aligned} \tag{8}$$

This problem is an instance of (1) with

$$f(\boldsymbol{x}, \boldsymbol{y}) = -xy_2, \quad g(\boldsymbol{x}, \boldsymbol{y}) = c_1 y_1 + (c_2 + x) y_2, \quad X = \mathbb{R}_+, \quad \text{and} \quad Y(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^2_+ : y_1 + y_2 = 1\}. \tag{9}$$

In the above problem, illustrated in Figure 1, x is the toll fee (which is to be determined by the leader) of the tolled path, and the fixed values c_1 and c_2 are the costs of traveling the untolled and tolled routes. Values y_1 and y_2 are the proportion of drivers that take the tolled and untolled ways. This optimistic formulation has the solution $(x, y_1, y_2) = (c_1 - c_2, 0, 1)$ (we assume that $c_1 > c_2$; otherwise, no driver would take the tolled route unless the toll is negative).

The pessimistic formulation of the same model, on the other hand, lacks a solution, because if $x < c_1 - c_2$, then $\mathcal{F}(\boldsymbol{x}) = \{(0,1)\}$ and the objective function value is $-x$. Therefore, increasing the toll fee x seems to improve the leader's profit. However, this is true only up to the point where $x = c_1 - c_2 > 0$, in which case $\mathcal{P}(\boldsymbol{x}) = \{(1,0)\}$ and we have a discontinuous increase of the leader's cost from $-x$ to 0. It is an example of the well-known fact that the pessimistic bilevel problem may fail to be solvable even when its optimistic counterpart does have a solution [3].

Notice that the optimistic formulation concludes that all drivers take the tolled path when the costs for traveling both routes are the same. In contrast, the pessimistic formulation assumes that all drivers take the untolled way when the expenses for traveling both paths are equal. From the follower's viewpoint, however, when $x = c_1 - c_2$, every feasible solution has an equivalent cost, and in practice, there is no reason why all drivers would choose the tolled route when all routes cost the same.

2 Limited Rationality

Let us start by defining the follower's ϵ-Rational Response Set $\mathcal{F}_\epsilon(\boldsymbol{x})$:

$$\mathcal{F}_\epsilon(\boldsymbol{x}) := \{\boldsymbol{y} \in Y(\boldsymbol{x}) : g(\boldsymbol{x},\boldsymbol{y}) \le g(\boldsymbol{x}) + \epsilon\}. \tag{10}$$

Wiesemann **et al.** [3] gives the following approximation to the pessimistic bilevel problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & W_{f,\epsilon}(\boldsymbol{x}) \\ \text{s.t.}: \quad & \boldsymbol{x} \in X, \end{aligned} \tag{11}$$

where

$$W_{f,\epsilon}(\boldsymbol{x}) := \sup_{\boldsymbol{y} \in \mathcal{F}_\epsilon(\boldsymbol{x})} f(\boldsymbol{x},\boldsymbol{y}). \tag{12}$$

According to Wiesemann **et al.** [3], these problems have, under mild hypothesis, solution for every $\epsilon > 0$. Moreover, the solution converges for $\epsilon \to 0$. We call this a "limited rationality" approach because less than optimal solutions are accepted for the follower.

Let us consider this formulation applied to the toll-setting problem above.

$$\mathcal{F}_\epsilon(\boldsymbol{x}) = Y(\boldsymbol{x}) \cap \begin{cases} \{\boldsymbol{y} : c_1 y_1 + (c_2 + x) y_2 \le (c_2 + x) + \epsilon\} & \text{if } c_2 + x \le c_1 \\ \{\boldsymbol{y} : c_1 y_1 + (c_2 + x) y_2 \le c_1 + \epsilon\} & \text{otherwise} \end{cases} \tag{13}$$

Parametrizing $Y(\boldsymbol{x})$ as $y_1 = \alpha$ and $y_2 = 1 - \alpha$, for $\alpha \in [0,1]$:

$$\tilde{\mathcal{F}}_\epsilon(\boldsymbol{x}) = \begin{cases} \{\alpha \in [0,1] : c_1\alpha + (c_2 + x)(1-\alpha) \le (c_2 + x) + \epsilon\} & \text{if } c_2 + x \le c_1 \\ \{\alpha \in [0,1] : c_1\alpha + (c_2 + x)(1-\alpha) \le c_1 + \epsilon\} & \text{otherwise} \end{cases} \tag{14}$$

Rearranjing:

$$\tilde{\mathcal{F}}_\epsilon(\boldsymbol{x}) = \begin{cases} 0 \le \alpha \le \min\left\{1, \frac{\epsilon}{c_1 - (c_2 + x)}\right\} & \text{if } c_2 + x < c_1 \\ 0 \le \alpha \le 1 & \text{if } c_2 + x = c_1 \\ \max\left\{0, \frac{c_1 - (c_2 + x) + \epsilon}{c_1 - (c_2 + x)}\right\} \le \alpha \le 1 & \text{otherwise.} \end{cases} \tag{15}$$

This can be simplified if we denote $\Delta = c_1 - (c_2 + x)$:

$$\tilde{\mathcal{F}}_\epsilon(\Delta) = \begin{cases} 0 \le \alpha \le \min\left\{1, \frac{\epsilon}{\Delta}\right\} & \text{if } \Delta > 0 \\ 0 \le \alpha \le 1 & \text{if } \Delta = 0 \\ \max\left\{0, \frac{\Delta + \epsilon}{\Delta}\right\} \le \alpha \le 1 & \text{otherwise.} \end{cases} \tag{16}$$

We now compute the objective function $W_{f,\epsilon}$ with respect to the variable Δ:

$$W_{f,\epsilon}(\Delta) = \sup_{\alpha \in \tilde{\mathcal{F}}_\epsilon(\Delta)} (1-\alpha)(\Delta - c_1 + c_2).$$

Notice that $\Delta - c_1 + c_2 = -x \leq 0$. Therefore, the supremum will be attained at $\alpha = \max \tilde{\mathcal{F}}_\epsilon(\Delta)$. That is,

$$W_{f,\epsilon}(\Delta) = \begin{cases} \left(1 - \frac{\epsilon}{\Delta}\right)(\Delta - c_1 + c_2) & \text{if } \Delta > \epsilon \\ 0 & \text{otherwise.} \end{cases}$$

The minimizer of $W_{f,\epsilon}(\Delta)$ is given by $\Delta_\epsilon^* = \sqrt{(c_1 - c_2)\epsilon}$. That is, the optimal toll for the ϵ-approximation of the original toll-setting problem is given by

$$x_\epsilon^* = (c_1 - c_2) - \sqrt{(c_1 - c_2)\epsilon}.$$

Therefore, if one follows the approach of Wiesemann **et al.**, the limiting solution of the ϵ-approximation turns out to be the same solution of the optimistic problem.

3 An Intermediate Formulation

A limitation of the optimistic, pessimistic and ϵ-approximate bilevel problems is that they model circumstances where the follower acts as single entity. The paper [2] introduces the intermediate formulation below, which overcomes this limitation:

$$\min_{\boldsymbol{x}} \quad E_f(\boldsymbol{x}) := \int_{\mathcal{F}(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\mu_{\boldsymbol{x}}(\boldsymbol{y}), \tag{17}$$

where $\mu_{\boldsymbol{x}}$ is an probability measure, which depends on $\boldsymbol{x}$. This intermediate formulation eliminates the assumption that the follower considers the leader's goals and gives arbitrary weights to all of the follower's rational responses. Problems (1) and (7) are determined by $f(\boldsymbol{x}, \boldsymbol{y})$, $g(\boldsymbol{x}, \boldsymbol{y})$, X, and $Y(\boldsymbol{x})$, whereas (17) also requires the definition of $\mu_{\boldsymbol{x}}$.

To analyze the intermediate problem for the toll-setting model, we can write the rational response set for this case explicitly:

$$\mathcal{F}(\boldsymbol{x}) = \begin{cases} (0,1) & \text{if } x < c_1 - c_2 \\ (\alpha, 1-\alpha) : \alpha \in [0,1] & \text{if } x = c_1 - c_2 \\ (1,0) & \text{if } x > c_1 - c_2. \end{cases} \tag{18}$$

Because $\mu_{\boldsymbol{x}}$ is a probability measure, the objective function of (17) has the form

$$E_f(\boldsymbol{x}) = \begin{cases} -x & \text{if } x < c_1 - c_2 \\ -q(c_1 - c_2) & \text{if } x = c_1 - c_2 \\ 0 & \text{if } x > c_1 - c_2, \end{cases} \tag{19}$$

where $q \in [0,1]$ is determined by the choice of the probability measure $\mu_{\boldsymbol{x}}$ made by the modeller. This is so because since $\mu_{\boldsymbol{x}}$ is a probability measure, we have

$$\int_{\mathcal{F}(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\mu_{\boldsymbol{x}}(\boldsymbol{y}) \in \left[\inf_{\boldsymbol{y} \in \mathcal{F}(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{y}), \sup_{\boldsymbol{y} \in \mathcal{F}(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{y}) \right] = [-x, 0] \tag{20}$$

and the integral equals $-x = -(c_1 - c_2)$ if and only if $\mu_{\boldsymbol{x}}$ is the Dirac measure of the point $\boldsymbol{y} = (0,1)$. A straightforward generalization of this argument justifies the terminology **intermediate**

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applied to (17) because its objective function can assume any value between the pessimistic and the optimistic cases, depending on how the measure $\mu_{\boldsymbol{x}}$ is selected.

Notice that if $q \in [0, 1)$ the problem has no solution, since the objective function can become arbitrarily close to $-(c_1 - c_2)$, but can never reach this value. If $q = 1$, then the solution is $x = c_1 - c_2$.

The analysis above shows that the intermediate formulation of [2] does not solve the issue of the non-existence of solutions in this simple toll-setting problem unless it matches the optimistic formulation (both the optimistic (1) and the pessimistic (7) formulations are special cases of the intermediate (17) case). This is because there is an abrupt change in the rational response set when $x = c_1 - c_2$. This discontinuous behavior causes mathematical difficulties, and it also does not model reality. In fact, the cost for travelling a route is not the same for all drivers for a variety of reasons and the costs proposed in the model are average costs. Because of that, in reality the probability that a given driver will take a path is likely to depend smoothly on the average cost.

4 A Generalized Approach

Our proposed approach is to use a formulation where the collective nature of the follower (the herd) is taken into account together with the limited rationality in a single formulation. The new formulation of the problem would then be

$$\min_{\boldsymbol{x}} \quad A_{f,\epsilon}(\boldsymbol{x}) := \int_{\mathcal{F}_\epsilon(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\mu_{\boldsymbol{x},\epsilon}(\boldsymbol{y}), \tag{21}$$

In the toll-setting problem above, using the uniform measure over $\mathcal{F}_\epsilon(\boldsymbol{x})$, we would have he objective function $A_{f,\epsilon}$:

$$\mathcal{A}_{f,\epsilon}(\boldsymbol{x}) = \begin{cases} -\frac{1}{\min\left\{1, \frac{\epsilon}{\Delta}\right\}} \int_0^{\min\left\{1, \frac{\epsilon}{\Delta}\right\}} x(1-\alpha)\mathrm{d}\alpha & \text{if } \Delta > 0 \\ -\int_0^1 x(1-\alpha)\mathrm{d}\alpha & \text{if } \Delta = 0 \\ -\frac{1}{1-\max\left\{0, \frac{\Delta+\epsilon}{\Delta}\right\}} \int_{\max\left\{0, \frac{\Delta+\epsilon}{\Delta}\right\}}^1 x(1-\alpha)\mathrm{d}\alpha & \text{otherwise} \end{cases} \tag{22}$$

Equivalently:

$$A_{f,\epsilon}(\boldsymbol{x}) = \begin{cases} -\frac{1}{\frac{\epsilon}{\Delta}} \int_0^{\frac{\epsilon}{\Delta}} x(1-\alpha)\mathrm{d}\alpha & \text{if } \Delta > \epsilon \\ -\int_0^1 x(1-\alpha)\mathrm{d}\alpha & \text{if } \epsilon \geq \Delta \geq -\epsilon \\ -\frac{1}{1-\frac{\Delta+\epsilon}{\Delta}} \int_{\frac{\Delta+\epsilon}{\Delta}}^1 x(1-\alpha)\mathrm{d}\alpha & \text{if } -\epsilon > \Delta. \end{cases} \tag{23}$$

In other words:

$$A_{f,\epsilon}(\boldsymbol{x}) = \begin{cases} -x\left(1 - \frac{\epsilon}{2\Delta}\right) & \text{if } \Delta > \epsilon \\ -x\frac{1}{2} & \text{if } \epsilon \geq \Delta \geq -\epsilon \\ x\frac{\epsilon}{2\Delta} & \text{if } -\epsilon > \Delta. \end{cases} \tag{24}$$

Although it is easy to minimize $A_{f,\epsilon}$ analytically, an illustration can be more clarifying. In Figure 2 we can see a comparison between our approach and Wiesemann's approach. Notice that both converge to the solution of the optimistic model as $\epsilon \to 0$. More interestingly, Wiesemann's approach never allows for a toll fee larger than $c_1 - c_2$ even given a large ϵ. This is so because his approach is still pessimistic, since although it takes limited rationality into consideration, it still assumes that there is a uniform antagonic behavior of all of the followers. Mallozzi's approach, on the other hand, takes into account nonuniform behavior, but assumes full rationality. Our aproach, however, takes both limited rationality and nonuniform behavior into account. This leads to the possibility of obtaining better solutions when there is a large "rationality gap" by setting the toll fee to values larger than $c_1 - c_2$.

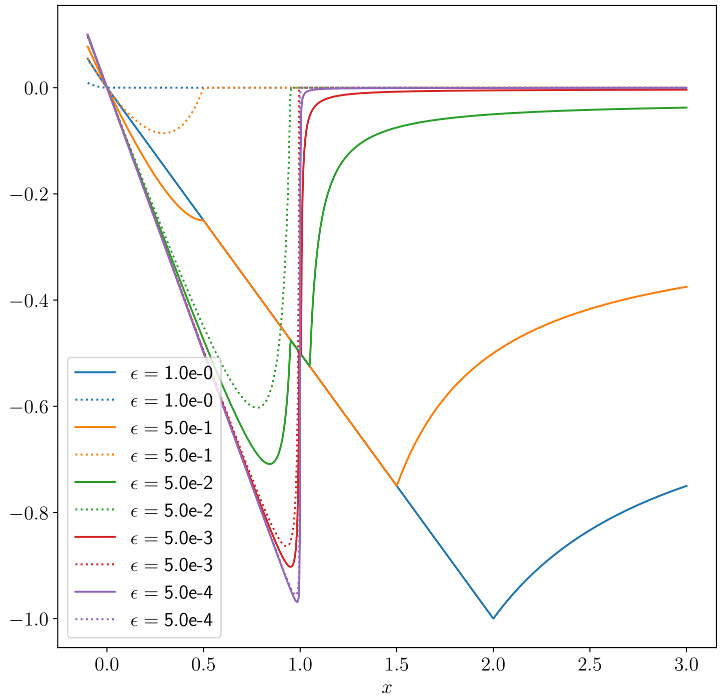

Figure 2: Comparison between the functions $A_{f,\epsilon}$ (using the uniform measure) and $W_{f,\epsilon}$ for the toll-setting problem with $c_1 = 2$ and $c_2 = 1$. Dotted lines depict the graphic of $W_{f,\epsilon}$, solid lines depict the graphic of $A_{f,\epsilon}$. Each color represents a different value for ϵ.. Source: produced by the author.

5 Concluding Remarks

We have presented a new generalization of the bilevel optimization problem and we have shown that it offers more generality than those currently available at the literature. This generality can be relevant, as shown in the simplest possible toll fee setting example. Future work includes studying the theoretical properties of the new approach as well as its application to more sophisticated problems.

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