Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 11, n. 1, 2025.

Trabalho apresentado no XLIII CNMAC, Centro de Convenções do Armação Resort - Porto de Galinhas - PE, 2024

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Homoclinic Orbits in the Modified Van der Pol System

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Abstract. This work studies a three non-linear ordinary differential equation system, depending on a set of eight parameters, which describes an economic model. The set of parameters are constrained in order to satisfy the Shilnikov Theorem, this is required when looking for the conditions for the existence of homoclinic orbits in a three-dimensional autonomous system.

Keywords. Homoclinic orbits, Van der Pol system, Shilnikov theorem, Non linear system

1 Introduction

Based on the Bouali's modification [3] of the two-dimensional Van der Pol oscillator [9], Amaral et al [1] proposed a modified version of the three-dimensional Van der Pol oscillator that takes into account the macroeconomic variable relationships. Their proposal is based on modifying the dynamics of the Gross Domestic Product, which is determined by the level of consumption, financed by household savings and flows of foreign capital. They ended up with a system of ordinary differential equations of three endogenous variables with eight positive parameters.

A preliminary analysis of stability and the Hopf bifurcation of Bouali's modification to Van der Pol's system was given in [10] and Pribylová [5] made both analytical and numerical analysis of several types of bifurcations.

In this work, we study the conditions for the existence of homoclinic orbits in the modified Van der Pol system. Homoclinic orbits are defined as trajectories which arrive at the same singular point when the time goes to $\pm\infty$, or in another words, the curves are connected at the same saddle point and is required to be a smooth one, i.e. the beginning and the end of the orbit should cross the same saddle point. We follow Bella et al's work [2] which describes in detail the steps to know under what conditions a steady point is connected to itself by a homoclinic orbit.

In Section 2, we present the model, derive its steady state conditions and study the local dynamics. The third section shows how the parameters are related in order to satisfy the Shilnikov theorem[7].

2 The Model

The modified Van der Pool ordinary differential equation is given by [1]

$$\begin{cases} \dot{x} = my + px(d - y^2) \\ \dot{y} = vy + wx + cz \\ \dot{z} = sx - ry, \end{cases} \tag{1}$$

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the set $\sigma = (m,\ p,\ d,\ c,\ s,\ r,\ v,\ w)$ of eight parameters modulate the behavior and dynamics of the system, the set lives inside $\Sigma \equiv \mathbb{R}^8_{++}$ and they represent the marginal propensity to saving, the fraction of capitalized profit, the GDP potential, the output-capital ratio, the capital inflow-saving ratio, the indebtedness factor, the marginal propensity to consume and the proportion of saving, respectively. The system $\mathcal{S}$ in Equation (1) has three steady states, the trivial one, $E_0 = (0, 0, 0)$ and the non-trivial ones $E_{1,2}$, which are given by

$$E_1 = \big(-\frac{r\varrho}{s}, -\varrho, \varrho\frac{rw + sv}{cs}\big) = -(x_*, y_*, z_*) = -E_2, \tag{2}$$

where $\varrho = \sqrt{d + \frac{ms}{pr}}$.

Let us denote by J, the Jacobian matrix of system evaluated at E_2. The eigenvalues of J are the solutions of the characteristic equation

$$\mathtt{det}(\lambda\mathtt{I} - J) = \lambda^3 - \mathtt{Tr}(J)\lambda^2 + \mathtt{B}(J)\lambda - \mathtt{Det}(J) \tag{3}$$

where $\mathtt{I}$ is the identity matrix. $\mathtt{Tr}(J)$ and $\mathtt{Det}(J)$ are Trace and Determinant of J, respectively. $\mathtt{B}(J)$ is the sum of principal minors of order 2. Explicit values are given bellow

$$\begin{aligned} \mathtt{Tr}(J) &= v - \frac{ms}{r},\ \ \mathtt{Det}(J) = -2scpx_*y_* \\ \mathtt{B}(J) &= cr - \frac{ms}{r}v - w(m - 2px_*y_*). \end{aligned} \tag{4}$$

Where x_* and y_* are the steady solutions for x and y in (2). It is straightforward to see that $m - 2px_*y_*$ is a negative number. The roots of polynomia in Equation (3) can be obtained by applying Cardano's formula and are given by

$$\lambda_1 = \eta = \gamma + \varphi + \frac{\mathtt{Tr}(J)}{3}, \quad \text{and} \quad \lambda_{2,3} = \tau \pm \omega i = -\frac{\gamma + \varphi}{2} + \frac{\mathtt{Tr}(J)}{3} \pm \frac{\gamma - \varphi}{2}\sqrt{3}i \tag{5}$$

with $\gamma = \sqrt[3]{-\frac{\delta}{2} + \sqrt{\Delta}}$, $\varphi = \sqrt[3]{-\frac{\delta}{2} - \sqrt{\Delta}}$, where

$$\Delta = \left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta_1}{3}\right)^3 \tag{6}$$

is the discriminant. Furthermore,

$$\delta_1 = \frac{3\mathtt{B}(J) - \mathtt{Tr}(J)^2}{3} \quad \text{and} \quad \delta = -\mathtt{Det}(J) - \frac{2}{27}\mathtt{Tr}(J)^3 + \frac{1}{3}\mathtt{Tr}(J)\mathtt{B}(J). \tag{7}$$

For this work, let us concentrate at the region Σ_1, which is parametrized by

$$\Sigma_1 = \left\{\sigma \in \Sigma : v \in (0, \frac{ms}{r}); \quad \frac{cr - w(m - 2px_*y_*)}{\frac{ms}{r}} < v\right\}. \tag{8}$$

3 Shilnikov Theorem in the Van der Pool system

In this section we show how the set of parameters in the Van der Pool system in equation (1) are constrained so that the model satisfies the requirements of the Shilnikov theorem [7]. For a pedagogical description on Shilnikov's Theorem, see [8] and for more elegant explanation [4]. We follow most of the algebraic computations done for the Lucas Model [2].

Theorem 3.1. *Given a third-order autonomous system in Equation* (1)*, let the eigenvalues* $\lambda_{1,2,3}$ *of the Jacobian at the equilibrium point to be the form* $\lambda_1 = \eta$ *and* $\lambda_{2,3} = \tau \pm \omega i$*, with* $\eta\tau < 0$

1. *The equilibrium point is a saddle focus and satisfies the saddle quantity* $\texttt{SQ} \equiv |\eta| - |\tau| > 0$
2. *There exist a homoclinic* Γ_0 *based at the saddle focus equilibrium point.*

The application of the theorem 3.1 to the Van der Pool system requires the following conditions to be met: *i)* In the case of hyperbolic saddle-focus equilibrium point, $\texttt{SQ}$ be positive; *ii)* in the hyperbolic saddle-focus point, with $\texttt{SQ} > 0$, there exists a homoclinic orbit, connecting the saddle-focus to itself. Bellow we show that system $\mathcal{S}$ supports the existence of a saddle-focus equilibrium point with $\texttt{SQ} > 0$.

For an equilibrium point to be a saddle-focus, the Jacobian matrix J must have a pair of roots with non-zero imaginary part, i.e. $\omega \neq 0$. Explicit expressions for roots $\lambda_{1,2,3}$ are given in Equation (5). Moreover, one eigenvalue should be negative $\eta < 0$ and the other two with positive real parts $\tau > 0$, i.e.

$$\eta = \gamma + \varphi + \frac{\texttt{Tr}(J)}{3} < 0 \;\; \texttt{and} \;\; \tau = -\frac{\gamma + \varphi}{2} + \frac{\texttt{Tr}(J)}{3} > 0 \tag{9}$$

and after some computations, the saddle quantity $\texttt{SQ}$ is given by

$$\texttt{SQ} = \frac{\frac{3}{2}(\gamma + \varphi)\left(\frac{2}{3}\texttt{Tr}(J) + \frac{\gamma+\varphi}{2}\right)}{\sqrt{\left(\gamma + \varphi + \frac{\texttt{Tr}(J)}{3}\right)^2} + \sqrt{\left(-\frac{\gamma+\varphi}{2} + \frac{\texttt{Tr}(J)}{3}\right)^2}} \tag{10}$$

and in order that $\texttt{SQ}$ to be positive, the following conditions must be fulfilled

$$\frac{2}{3}\texttt{Tr}(J) + \frac{\gamma + \varphi}{2} < 0 \;\; \texttt{and} \;\; \gamma + \varphi < 0. \tag{11}$$

The last inequality in Equation(11) is satisfied for any $\delta > 0$[3]. Moreover, $\delta \in \left(\frac{2\sqrt{3}}{9}\sqrt{-\delta_1^3}, \infty\right)$

and this fact implies that δ_1 is negative in Σ_1 given in the Equation (8). Explicit calculation of δ_1 show this argument

$$3\delta_1 = 3\left(cr - w(m - 2px_*y_*) - \frac{ms}{r}v\right) - \left(v - \frac{ms}{r}\right)^2 < 0. \tag{12}$$

where, the first term in the right-hand side is negative inside of Σ_1. The region $\Phi = 0$ where both the discriminant Δ and the saddle quantity $\texttt{SQ}$ vanishes was computed explicitly in terms of $\texttt{B}(J)$ and $\texttt{Tr}(J)$ [2]. And, is given by $\texttt{B}(J) + \texttt{Tr}^2(J) = 0$. $\texttt{B}(J)$ and $\texttt{Tr}(J)$ in terms of the set σ are given in Equation (4). Therefore, the surface $\Phi = 0$ in terms of the set σ is given by

$$\hat{v}^2 - 3\frac{ms}{r}\hat{v} + \left(\frac{ms}{r}\right)^2 + cr + w\left(m + \frac{2prd}{s}\right) = 0. \tag{13}$$

From equation (13), we can see that $\hat{v}$ admits two values, $\hat{v}_\pm$. So, we must choose some v such that

$$v < \hat{v}_- \quad \text{or} \quad v > \hat{v}_+. \tag{14}$$

Moreover, from (13) we obtain equation for the c parameter

$$c = \frac{1}{r}\left\{\frac{3ms}{r}\hat{v} - \left(\frac{ms}{r}\right)^2 - \hat{v}^2 - \left(m + \frac{2prd}{s}\right)w\right\}. \tag{15}$$

[3] Since $\gamma^3 + \varphi^3 = (\gamma + \varphi)(\gamma^2 - \gamma\varphi + \varphi^2) = -\delta$ and $\gamma\varphi > 0$

DOI: 10.5540/03.2025.011.01.0405

From $\Delta = 0$, we conclude that $\mathtt{Det}(J) = \frac{5}{27}\mathtt{Tr}(J)^3$, in terms of the parameters we get

$$w = \frac{\frac{3ms}{r}\hat{v} - \left(\frac{ms}{r}\right)^2 - \hat{v}^2 + \frac{5}{54}\frac{\left(\hat{v}-\frac{ms}{r}\right)^3}{pd+\frac{ms}{r}}}{\frac{r}{s}\left(\frac{ms}{r} + 2pd\right)}. \tag{16}$$

These facts show that $\Sigma_1 \neq \emptyset$.

Example 1. Consider $(m,\ p,\ d,\ c,\ s,\ r,\ w) = (0.02,\ 0.4,\ 1.0,\ 0.5,\ 10,\ 0.1,\ 0.1)$. From equation (13) we have $\hat{v}_- \approx 0.7758$ and $\hat{v}_+ \approx 5.2242$. Since $\hat{v}_- < \frac{ms}{r}$ and $\frac{cr+w(m+2\frac{prd}{s})}{\frac{ms}{r}} < \hat{v}_-$, $(m,\ p,\ d,\ c,\ s,\ r, \hat{v}_-,\ w) \subset \Sigma_1$. Set therefore $(m,\ p,\ d,\ c,\ s,\ r,\ v,\ w) = (0.02,\ 0.4,\ 1.0,\ 0.5,\ 10,\ 0.1,\ 0.05,\ 0.1)$. Then

$$\lambda_1 \approx -2.0314 \quad \text{and} \quad \lambda_{2,3} \approx 0.0407 \pm 0.3413i$$

with $\mathtt{SQ} = |\eta| - |\tau| \approx 1.9907 > 0$

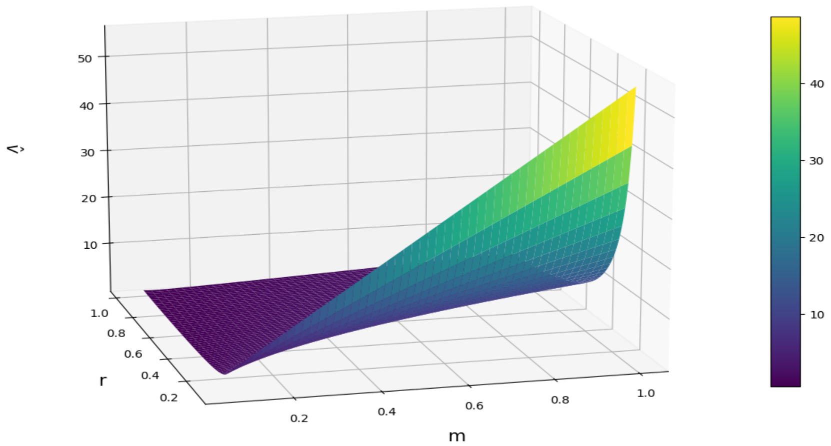

Figure 1: The bifurcation surface shows $(m,\ r,\ \hat{v}_-)$ combinations such that $\Delta = 0 = \mathtt{SQ}$. Source: Own work

4 Homoclinic Orbits

In the previous section, we set up the region in the parameter space at which E_2 is a saddle-focus equilibrium with positive saddle-quantity, i.e. $\mathtt{SQ} > 0$. The method of the undetermined coefficients [6] can be used to show that our system in equation (1) admits homoclinic solutions. The implementation of the method requires to put the system $\mathcal{S}$ into an appropriate form to work with. To do this, let us translate the equilibrium point to the origin, by assuming

$$\tilde{x} = x - x_*, \quad \tilde{y} = y - y_*, \quad \tilde{z} = z - z_*, \tag{17}$$

and let $\mathbf{T}$ be a matrix of the eigenvectors associated wit the structure of eigenvalues of J.

$$JU = \tau U - \omega V, \quad JV = \tau V + \omega U \quad \texttt{and} \quad JZ = \eta Z. \tag{18}$$

Then, it is possible to put the system $\mathcal{S}$ in to the following Jordan normal form

$$\dot{W} = \mathbf{T^{-1}} J \mathbf{T} W + \mathbf{T^{-1}} \tilde{f}(\mathbf{T}W), \tag{19}$$

given the associated change in coordinates

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \mathbf{T} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \quad \texttt{where} \quad \mathbf{T} = [U, V, Z] \tag{20}$$

Therefore, we obtain the following system

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{bmatrix} \tau & \omega & 0 \\ -\omega & \tau & 0 \\ 0 & 0 & \eta \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} F_1(w_1, w_2, w_3) \\ F_2(w_1, w_2, w_3) \\ F_3(w_1, w_2, w_3) \end{pmatrix} \tag{21}$$

with $F_i(w_1, w_2, w_3) = F_{ia} w_1 w_2 + F_{ib} w_1 w_3 + F_{ic} w_2 w_3 + F_{id} w_1^2 + F_{ie} w_2^2 + F_{if} w_3^2$ and $i = 1, 2, 3$ The F coefficients are combinations of the original parameters of the model in Equation (1), also depending on the values of constants c_2 and c_3 arising when computing the eigenbasis U, V, Z.

Lema 4.1. *Denote the set* $\Upsilon \equiv \{\sigma \in \Sigma_1 : v \in (0, \hat{v})$ *such that the system* $\mathcal{S}$ *has a homoclinic orbit*$\}$. *Then* $\Upsilon \neq \emptyset$

Proof. By using the method of the undetermined coefficients to the system $\mathcal{S}$ requires that the expression

$$\xi = -\frac{F_{3f}}{\eta} \frac{4\eta^2 - 4\eta\tau + \omega^2 + \tau^2}{(2\eta - \tau + \omega)F_{2f} + (2\eta - \tau - \omega)F_{1f}} \left(\zeta + \phi + \frac{\mathcal{N}_1}{9\omega^4 + \tau^4 + 10\omega^2\tau^2} \right) + \frac{\mathcal{N}_2}{4\omega^2 + (\eta - 2\tau)^2}, \tag{22}$$

be satisfied for homoclinic solutions to exist. In (22), $(\xi,\ \phi,\ \zeta)$ is a triplet of arbitrary constants. $\mathcal{N}_1$ and $\mathcal{N}_2$ are given by

$$\begin{aligned} \mathcal{N}_1 &= (\phi^2 - \zeta^2)\Big\{ L_1 F_{1a} + L_2 F_{2a} + L_3(F_{1d} - F_{1e}) + L_4(F_{2d} - F_{2e}) \Big\} \\ &\quad + 2\phi\zeta \Big\{ - L_3 F_{1a} - L_4 F_{2a} + L_1(F_{1d} - F_{1e}) + L_2(F_{2d} - F_{2e}) \Big\} \end{aligned} \tag{23}$$

$$\begin{aligned} &\mathcal{N}_2 = (\phi^2 - \zeta^2)\Big\{ L_5 F_{3a} + L_6(F_{3d} - F_{3e}) \Big\} + 2\phi\zeta \Big\{ - L_6 F_{3a} + L_5(F_{3d} - F_{3e}) \Big\} \texttt{ with} \\ &L_1 = -2\omega(3\omega^2 - \tau^2) + 4\omega\tau(\omega - \tau), \quad L_2 = -2\omega(3\omega^2 - \tau^2) - 4\omega\tau(\omega + \tau), \quad L_5 = -2\omega \\ &L_3 = (\tau - \omega)(3\omega^2 - \tau^2) - 8\omega^2\tau, \quad L_4 = (\tau + \omega)(3\omega^2 - \tau^2) - 8\omega^2\tau, \quad L_6 = \eta - 2\tau. \end{aligned} \tag{24}$$

The existence of homoclinic loop asymptotic to the saddle-focus equilibrium point requires that $v \in (0, \hat{v})$, with $\sigma \in \Sigma_1$, at which (22) is satisfied. Therefore $\Upsilon \neq \emptyset$. □

We provide bellow an illustrative example implying the existence of as homoclinic solution of system in equation (1).

Example 2 Let $(m,\ p,\ d,\ c,\ s,\ r,\ w) = (0.02,\ 0.4,\ 1.0,\ 0.5,\ 10,\ 0.1,\ 0.1)$ as in Example 1. This choice implies $\hat{v} = 0.7758$. Set $(c_2, c_3) = (0.001, 0.2)$ and assume $\phi = \zeta$. Then Equation (22) gives rise to the surface in (ξ, v, ζ) coordinates(see Figure 2)

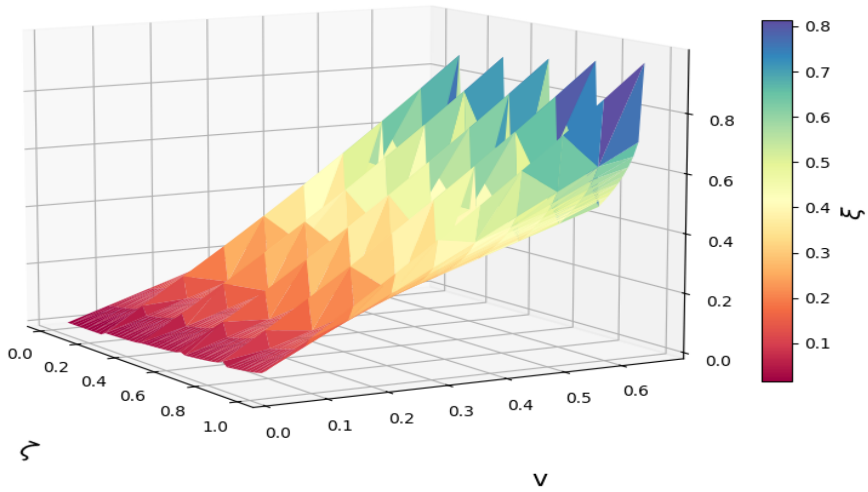

Figure 2: Combinations of (ξ, v, ζ) for the family of homoclinic solutions. Source: Own work

5 Final Considerations

This project observed an economic model using dynamics, non-linear equations developing a dynamic system. The main objective was to identify in it a region with a certain stability by finding a set of parameters that would stabilize the system. Dynamical systems such as the one presented in the economic model are characterized by high sensitivity to initial conditions, self-similarity and fractals. The high sensitivity to initial conditions gives the non-linear system a certain instability, and this instability in the system out comes in sensitivity to disturbances and errors, generating results that are not expected. This may be the subject of future work on the project.

Acknowledgments

We are thankful for the support to CNMAC2024's organizers and to SBMAC for the opportunity to take part in this Congress. The authors would like to thank Prof. Dr. Fernando F. Ferreira for suggesting us the problem and the anonymous referees for reading the manuscript and for suggesting observations.

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