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Primal-dual Inequalities for the Approximate Maximum-cut Value of Graphs in Homogeneous Coherent Configurations

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Let G be a simple undirected graph with vertex set V and edge set E. We say that a subset K of V is a clique if its induced subgraph is a complete graph on $|K|$ vertices, and similarly we say that a subset C of V is a coclique if its induced subgraph has no edges. Denote by $\omega(G)$ and $\alpha(G)$ the size of the largest clique and the largest coclique of G, respectively. In general, computing these two parameters for a given graph is NP-hard, and there doesn't seem to be any correlation between the existence of large cliques and the nonexistence of large cocliques for an arbitrary graph, and vice-versa. However, for graphs with enough structural regularity, one can find a strong relationship between such parameters, known as the clique-coclique inequality [3]. In this work, we seek to study ways of generalizing such inequality for primal-dual formulations of problems related to the maximum-cut of highly regular graphs.

Let $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices with complex-valued entries. We say that a subset $\mathcal{A}$ of such set is an algebra if it is a $\mathbb{C}$-vector subspace that is also a ring with respect to matrix multiplication. If this set is also closed with respect to the conjugate-transpose operation, we call it a $*$-algebra. By a *coherent algebra* [6] we mean a $*$-algebra that contains the identity matrix I and that is also a unitary ring with respect to Schur multiplication, i.e., it contains the all-ones matrix J and it is closed for entry-wise multiplication. It can be shown [5] that such algebras have a basis of 01 matrices whose sum equals J and such basis is called a *coherent configuration.* If I belongs to this basis, we call it *homogeneous* and if the matrices in the basis also commute, we call it an *association scheme* that generates a *Bose-Mesner algebra.*

By highly regular graphs, we mean graphs whose adjacency matrix belongs to a homogeneous coherent algebra, i.e., can be expressed as a sum of matrices in a coherent configuration. The aforementioned clique-coclique bound states that, for a graph G in a Bose-Mesner algebra, we have

$$\alpha(G)\omega(G) \leq |V|, \tag{1}$$

which intuitively shows that the existence of large cliques implies the nonexistence of large cocliques for such graphs, and vice-versa.

Given a graph G, we let A denote its adjacency matrix and $\overline{G}$ denote its complement, and for a given matrix X, we write $X \succcurlyeq 0$ if the matrix is positive semidefinite. In [2], the authors generalize the clique-coclique bound for graphs in homogeneous coherent configurations, i.e., the commutativity property is shown to not be required for such a bound. They also prove a similar result for the Lovàsz Theta parameter [8] of a graph, defined as

$$\vartheta(G) = \max\{\langle J, X\rangle | X \circ A(G) = 0, \mathrm{tr}(X) = 1, X \succcurlyeq 0\}, \tag{2}$$

where $\langle J, X\rangle := \mathrm{Re}(\mathrm{tr}(JX^*))$. The Theta parameter is closely related to the aforementioned graph parameters; however it can be computed efficiently, thus being of great theoretical and practical

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interest. The authors show that, for graphs in homogeneous coherent configurations, the following holds:

$$\vartheta(G)\vartheta(\overline{G}) = |V|. \tag{3}$$

Two main tools are used to prove this result: (i) the fact that any $*$-algebra is semisimple [7], i.e., it can be expressed as a finite direct sum of simple subalgebras, and (ii) the orthogonal projection of any positive semidefinite operator onto a $*$-algebra remains positive semidefinite [1].

In this work, we seek to study how to adapt these techniques in order to obtain similar results for the semidefinite programming (SDP) approximation of the maximum-cut problem and its dual, and to other graph-related parameters introduced in [9] for graphs in homogeneous coherent configurations. One of our partial results is that for a graph which is 1-walk-regular, the optimum of the Goemans-Williamson [4] SDP relaxation for the maximum-cut is easily expressed in terms of the smallest eigenvalue of the graph.

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