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Study on the Performance of SPC Product Codes

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The single parity-check (SPC) code is a very popular MDS error detection code, since it is very easy to implement [5]. One bit is appended to an information sequence of length $n-1$, such that the resultant codeword has an even number of ones. Two SPC codes can be used jointly to obtain an SPC product code, which has a minimum distance of 4. Therefore, it can recover all erasure patterns with one, two or three erasures over the erasure channel [3]. However, in some special cases, patterns of up to $2n-1$ erasures can be corrected. Furthermore, a codeword of length n^2 can be represented by an erasure pattern of size $n \times n$, where the only information considered is the position of the erasures (see for example, Figure 1(a)).

In [5], authors proposed a tight upper bound of the post-decoding erasure rate of the SPC product code. This approach was based on observing the structure of the erasure patterns, which makes it possible to classify them into correctable and uncorrectable patterns. The pattern in Figure 1(a) is uncorrectable, since it is not possible to correct the subpattern in gray. Later, in [1], the authors performed a counting method to obtain the number of uncorrectable erasure patterns with $4, 5, 6, 7, 8, 2n-3, 2n-2$ and $2n-1$ erasures. After that, in [2], the authors represented each erasure pattern by a bipartite graph with n nodes in each vertex class and the same number of edges as erasures. In fact, there exists an edge joining two vertices if there is an erasure in the corresponding position of the equivalent erasure pattern. It is possible to check that patterns that contain uncorrectable subpatterns are represented by bipartite graphs with cycles (see for instance Figure 1(b)). Then, the problem of counting uncorrectable erasure patterns can be seen as a problem of counting bipartite graphs with cycles. At the same time, a bipartite graph can be represented by a binary matrix where a 1 appears if two vertices of different vertex classes are connected, that is, the bi-adjacency matrix [4]. In Figure 1(c) one can observe the bi-adjacency matrix of the graph in Figure 1(b). Thus, the counting problem can be seen as counting binary matrices with specific properties. In [2], the authors counted binary bi-adjacency matrices of graphs with cycles to provide a lower and an upper bound on the number of uncorrectable erasure patterns (bipartite graphs having no vertex of degree 1). Notice that the determination of the exact number of uncorrectable (or correctable) erasure patterns is an open problem and few results have been discovered up to now.

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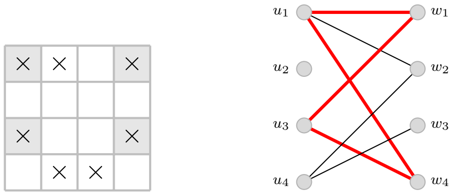

$$
\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}
\begin{array}{c} \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}
$$

(a) Pattern with 7 erasures (b) Bipartite graph (c) Bi-adjacency matrix

Figure 1: Erasure pattern, its bipartite graph and its bi-adjacency matrix. Source: The authors.

Each pattern can be represented by a binary word $\boldsymbol{x}$ of length n^2 as follows. For $i, j = 1, 2, \ldots, n$, we have

$$
x_{n(i-1)+j} = \begin{cases} 1 \text{ if there is an erasure in position } (i, j) \text{ of the pattern,} \\ 0 \text{ if there is no erasure in position } (i, j) \text{ of the pattern.} \end{cases} \tag{1}
$$

For instance, the pattern in Figure 1(a), can be represented by $[1\ 1\ 0\ 1|0\ 0\ 0\ 0|1\ 0\ 0\ 1|0\ 1\ 1\ 0]$, coinciding with the rows of the bi-adjacency matrix. Now, let H be the parity-check matrix of the SPC product code and consider $\boldsymbol{x}$ a word that represents an uncorrectable pattern such that the positions of the ones are in $\mathcal{X}$. It is possible to prove that the set $\{h_i \mid i \in \mathcal{X}\}$ of columns of H, where h_i is the i-th column, is is a linearly dependent set. Moreover, the set $\{h_i \mid i \notin \mathcal{X}\}$ of columns of H, is is a linearly independent set. This implies that all codewords of $\mathcal{C}$ represent uncorrectable strict patterns. In other words, the number of codewords of weight t is a lower bound on the number of uncorrectable patterns with t erasures. In this work, we study the weight enumerator polynomial of the SPC product code to derive bounds on the number of uncorrectable erasure patterns, thereby investigating the performance of the code.

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