Aggregating fuzzy e-Xor connectives based on the OWA-operator

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Abstrac: *The main contribution of this paper is concerned with the aggregating fuzzy e-Xor connectives based on the OWA-operator focusing on the fuzzy e-Xor implication class and related dual construction. The paper formally states that such aggregation operator preserves the main properties of fuzzy connectives in corresponding classes.*

Keywords: *fuzzy logic, aggregation function, OWA-operator, fuzzy Xor, fuzzy Xor implications*

1 Introduction

In Fuzzy Logic, the "fuzzy exclusive or" (Xor) and its dual construction – the XNor connective are extensively studied, e.g.[9, 8, 10, 11] and [12], related to applications and basic concepts which are concerned with knowledge based systems, either from image processing to decision making or from pattern recognition to machine learning.

By aggregating a family of fuzzy X(N)or connectives making use of the ordered weighted average aggregation operator (OWA-operator), this paper deals with an extension of fuzzy X(N)or named as the fundamental class of representable fuzzy subX(N)or and corresponding fuzzy implications.

Triangular sub(co)norms and strong fuzzy negations are introduced in[12] introducing the aggregation of fuzzy (S,N)- and R-implications by the arithmetic media.

The results achieved in [1, 2] are here extended into a discussion on the fuzzy subXor class, mainly related to relax boundary conditions in the definition of the fuzzy Xor class. We also investigated under which conditions such class is preserved by n-ary aggregation functions.

Instances of the fuzzy subXor class are reported but they are restricted to the fuzzy e-subXor connectives, referred as $\mathsf{E}_{S,T}$. Thus, an e-Xor connective is obtained as a difference between a t-subconorm S and a t-subnorm T. By taking the algebraic sum S_P and its mutual dual construction, the product t-norm T_P, the main algebraic properties, dual construction and graphical representation of the E_{S_P,T_P} are discussed.

This paper presents the main results obtained from the OWA-operator performed over a finite family of e-subXor connectives, introducing the operator $\mathcal{E}_\mathcal{A}$, called $(\mathcal{A}, \mathcal{E})$-**operator**. Its dual construction $\mathcal{D}_\mathcal{A}$, called $(\mathcal{A}, \mathcal{D})$-**operator** is also considered including algebraic properties. Additionally, a new fuzzy sub(co)implication is obtained from the OWA-operator and a finite family in the fuzzy e-subX(N)or (co)implication class.

This paper is organized as follows. The next section describes the basic concepts of fuzzy connectives. In Section 3, the fuzzy e-Xor connective, dual construction and related e-Xor (co)implication class are discussed. As our main contribution, Section 4 shows the aggregating fuzzy e-X(N)or connectives and related (co)implications. Finally, conclusions are drawn in Section 5.

2 Preliminaries

In this section, basic definitions related to the fuzzy connectives: fuzzy negation and triangular (co)norms (t-(co)norms) are considered. For that, let $U = [0, 1]$ be the unit interval.

2.1 Fuzzy Negations

A function $N : U \to U$ is a *fuzzy negation* if

N1: $N(0) = 1$ and $N(1) = 0$;   **N2:** If $x \geq y$ then $N(x) \leq N(y), \forall x, y \in U$.

If N also verifies the involutive property, it is called a *strong fuzzy negation* (SFN) [6]:

N3: $N(N(x)) = x, \forall x \in U$.

The most usual SFN is $N_S(x) = 1 - x$. Additionally, let N be a fuzzy negation on U and $f : U^n \to U$ be a real function. The *N-dual function* of f is given by :

$$f_N(x_1, \ldots, x_n) = N(f(N(x_1), \ldots, N(x_n))). \tag{1}$$

When N is involutive, $(f_N)_N = f$, that is the N-dual function of f_N coincides with f. In addition, if $f = f_N$ then it is clear that f is a self-dual function [4].

2.2 Aggregation Operators

In [13, Definition 2], an n-ary *aggregation* is a function $A \colon U^n \to U$ demanding, for all $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{y} = (y_1, y_2, \ldots, y_n) \in U^n$, the following conditions:

A1: $A(\vec{0}) = A(0, 0, \ldots, 0) = 0$ and $A(\vec{1}) = A(1, 1, \ldots, 1) = 1$;

A2: If $\vec{x} \leq \vec{y}$ then $A(\vec{x}) \leq A(\vec{y})$;

A3: $A(\vec{x_\sigma}) = A(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}) = A(\vec{x})$ when $\sigma \colon \mathbb{N}^n \to \mathbb{N}^n$ is a σ-permutation.

Some extra usual properties for aggregation functions are the following:

A3: Symmetry $A(\vec{x_\sigma}) = A(x_{\sigma_1}, \ldots, x_{\sigma_n}) = A(\vec{x})$, for all σ-permutation;

A4: Compensation $\min_{i=1}^{n}(x_i) \leq A(\vec{x}) \leq \max_{i=1}^{n}(x_i)$;

A5: Idempotency $A(x, x, \ldots, x) = x$, for all $x \in U$;

Proposition 1 *Let $\sigma \colon \mathbb{N}_n \to \mathbb{N}_n$, with $\mathbb{N}_n = \{1, \ldots, n\}$, be a permutation ordering the elements: $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$. Let $w_1, \ldots, w_n$ be non negative weights ($w_i \geq 0$) such that $\sum_{i=1}^{n} w_i = 1$. The operator $\mathcal{A} \colon U^n \to U$ which is named ordered weighted average aggregation operator (OWA-operator) and given as*

$$\mathcal{A}(\vec{x}) = \sum_{i=1}^{n} w_i x_{\sigma(i)}, \tag{2}$$

verifies property **Ak***, for* **k** $\in \{3, 4, 5\}$.

Proof: Straightforward. ▲

Proposition 2 *Let $\sigma \colon \mathbb{N}_n \to \mathbb{N}_n$ be a σ-permutation and $w_1, \ldots, w_n$ be non negative weights ($w_i \geq 0$) such that $\sum_{i=1}^{n} w_i = 1$. Then, $0 \leq \sum_{i=1}^{n} \frac{w_i}{\sigma(i)} \leq 1$.*

Proof: For all $x_{\sigma(i)} \in U$, it is immediate that $1 = \sum_{i=1}^{n} w_i \geq \sum_{i=1}^{n} \frac{w_i}{\sigma(i)} \geq 0$. ▲

2.3 Triangular Norms and Conorms

According with [6], a **triangular sub(co)norm** (t-sub(co)norm) is a binary aggregation function $T(S) \colon U^2 \to U$ such that, for all $x, y, z \in U$, the following holds:

T0: $T(x, y) \leq \min(x, y)$;   **S0:** $S(x, y) \geq \max(x, y)$

and also verifying the commutativity, associativity and monotonicity properties which are, respectively, given by the next three expressions:

T1: $T(x, y) = T(y, x)$;   **S1:** $S(x, y) = S(y, x)$;

T2: $T(x, T(y, z)) = T(T(x, y), z)$;   **S2:** $S(x, S(y, z)) = S(S(x, y), z)$;

T3: $T(x, z) \leq T(y, z)$, if $x \leq y$;   **S3:** $S(x, z) \leq S(y, z)$, if $x \leq y$.

Additionally, a **triangular (co)norm** (t-(co)norm) is -sub(co)norm also satisfying the property:

T4: $T(x, 1) = x$;   **S4:** $S(x, 0) = x$.

Proposition 3 *For $i \geq 1$ and $x, y \in U$, a function $T_i\,(S_i) : U^2 \to U$ is a t-sub(co)norm given by*

$$T_i(x, y) = \frac{1}{i}xy \qquad \left(S_i(x, y) = 1 - \frac{1}{i}(1 - x - y + xy)\right). \tag{3}$$

Proof: Straightforward.   ▲

When $i = 1$, by Eq.(3)a and Eq.(3)b we obtain the product t-norm and probabilistic sum, which are mutual N_S-dual t-(co)norms, respectively given by

$$T_P(x, y) = xy \qquad (S_P(x, y) = x + y - xy). \tag{4}$$

3 Fuzzy e-Xor and Dual Construction

This section is based on the X(N)or connective concepts as described in [2].

Definition 1 *A function $\mathsf{E}(\mathsf{D}) : U^2 \to U$ is a **fuzzy subX(N)or** if it satisfies, for all $x, y \in U$, the following properties:*

E0(i): $\mathsf{E}(1, 0)=1$;   **D0(i):** $\mathsf{D}(0, 1)=0$;

E1: $\mathsf{E}(x, y) = \mathsf{E}(y, x)$;   **D1:** $\mathsf{D}(x, y) = \mathsf{D}(y, x)$;

E2(i): *If $x \leq y$ then $\mathsf{E}(0, x) \leq \mathsf{E}(0, y)$;*   **D2(i):** *If $x \leq y$ then $\mathsf{D}(0, x) \geq \mathsf{D}(0, y)$;*

E2(ii): *If $x \leq y$ then $\mathsf{E}(1, x) \geq \mathsf{E}(1, y)$;*   **D2(ii):** *If $x \leq y$ then $\mathsf{D}(1, x) \leq \mathsf{D}(y, 1)$.*

Definition 2 *A fuzzy subX(N)or satisfying:*

E0(ii): $\mathsf{E}(1, 1)=\mathsf{E}(0, 0)=0$; *(D0(ii):* $\mathsf{D}(1, 1)=\mathsf{D}(0, 0)=1$)

*is called a **fuzzy exclusive or (Xor) (fuzzy exclusive nor (XNor)).**

Some extra reasonable properties can be considered for the fuzzy Xor connective (see [2] for a more thorough discussion):

E3: $\mathsf{E}(x, x) \neq 1$;

E4: If $\mathsf{E}(x, y) = 1$ then ($x = 1$ and $y = 0$) or ($x = 0$ and $y = 1$).

By considering any fuzzy negation, another generalization of this fuzzy Xor E_K is introduced in [1] and reported below. In [1, Prop. 3.6], when T, S and N are a t-norm, a t-conorm and a fuzzy negation, respectively, for all $x, y \in U$, the function $E_{S,T}(D_{T,S}) : U^2 \to U$, defined by

$$E_{S,T}(x, y) = S(x, y) - T(x, y) \qquad (D_{T,S}(x, y) = N(N(T(x, y)) - N(S(x, y)))) \qquad (5)$$

is a fuzzy Xor (XNor) connective called in this paper as **fuzzy e-X(N)or connective**. Moreover, by [1, Prop. 3.7], it holds that $N_{E_{S,T}}(x) = E_{S,T}(1, x) = N(x)$.

Proposition 4 *[1, Prop. 2] When $T = T_P$, $S = S_P$ and $N = N_S$ then , for all $x, y \in U$, it holds that*

$$E_{S_P,T_P}(x, y) = x + y - 2xy \qquad (D_{T_P,S_P}(x, y) = 1 - x - y + 2xy), \quad \forall x, y \in U. \qquad (6)$$

Proposition 5 *[1, Prop. 3] The fuzzy e-Xor connective E_{S_P,T_P} satisfies **E3** and **E4**.*

3.1 Fuzzy e-X(N)or (co)implication class

A **fuzzy subimplicator** $I : U^2 \to U$ satisfies the conditions:

I0 : $I(1, 1) = I(0, 1) = I(0, 0) = 1$;

A fuzzy subimplicator $I : U^2 \to U$ also satisfying the boundary condition:

I1 : $I(1, 0) = 0$;

is called **fuzzy implicator**. And, a fuzzy (sub)implicator I verifying:

I2 : If $x \le z$ then $I(x, y) \ge I(z, y)$ (left antitonicity);

I3 : If $y \le z$ then $I(x, y) \le I(x, z)$ (right isotonicity);

I4 : $I(0, y) = 1$ (left boundary property);

is called a **fuzzy (sub)implication** [**?**, Def. 6][5].

There exist many classes of (co)implication functions (see, e.g., [3] and [**?**]). We consider in this paper the e-Xor (eXNor-coimplication) is studied [2], based on the equivalence of the classical logic: $\alpha \to \beta \equiv \neg\alpha \vee (\neg\alpha \oplus \beta)$.

Let $T(S)$, N and $E(D)$ be a t-(co)norm, a fuzzy negation and a fuzzy Xor (XNor), respectively. According to [2], the function $I_{S,N,E}(J_{D,T,N}) : U^2 \to U$, given by

$$I_{E,S,N}(x, y) = E(x, S(N(x), N(y))), \quad (J_{D,T,N}(x, y) = D(x, T(N(x), N(y)))). \qquad (7)$$

is a fuzzy (co)implication, called *e-Xor implication (e-XNor coimplication)*. Moreover, $N_I : U^2 \to U$, $N_I(x) = I(x, 0)$ is the fuzzy negation underlying I.

Proposition 6 *When $S = S_P$, $N = N_S$ and $T = T_P$, $\forall x, y \in U$, then*

$$I_{E_{S_P,T_P},S_P,N_S}(x, y) = 1 - x - xy + 2x^2y \; \left(J_{D_{T_P,S_P},T_P,N_S}(x, y) = y - 2x + xy + 2x^2 - 2x^2y\right) \quad (8)$$

*satisfies **Ik** (**Jk**) for $k \in \{1, 2, 3\}$.*

4 Aggregating Fuzzy Xor Conectives

Definition 3 *[12, Prop. 5.1] The function $\mathcal{F}_A : U^k \to U$ called as $(A, \mathcal{F})$-operator on U is given by:*

$$\mathcal{F}_A(x_1, \ldots, x_k) = A(F_1(x_1, \ldots, x_k), \ldots, F_n(x_1, \ldots, x_k)). \qquad (9)$$

4.1 Aggregating fuzzy t-subnorms and t-subconorms

Proposition 7 *[12, Proposition 6.1] Let A be an n-ary aggregation operator and $\{T_1, T_2, \ldots . T_n\}$ and $\{S_1, S_2, \ldots, S_n\}$ be finite sets of t-subconomrs and t-subnorms both verifying distributivity and generalized associativity, respectively. The functions $\mathcal{T}_A, \mathcal{S}_A \colon U^k \to U$ called as $(A, \mathcal{T})$- and $(A, \mathcal{S})$- **operators on** U and given by Eq.(9) are fuzzy t-subnorm and fuzzy t-subconorm, respectively.*

4.2 Aggregating Fuzzy e-X(N)or Connectives

In this section, focusing on the fuzzy e-XNor class, a methodology is presented in order to obtain a new fuzzy e-subX(N)or connective from the OWA-operator and a finite subset of fuzzy e-subX(N)or connectives.

Proposition 8 *The function $\mathsf{E}_i(\mathsf{D}_i) : U^2 \to U$ given by:*

$$\mathsf{E}_i(x,y) = 1 - \frac{1}{i}(1 - x - y + 2xy) \left(\mathsf{D}_i(x,y) = \frac{1}{i}(1 - x - y + 2xy)\right), \forall x, y \in U, \qquad (10)$$

whenever i is a rational index such that $i \geq 1$, verifies **Ek (Dk)** *for* $\mathbf{k} \in \{0(i), 1, 2(i), 2(ii), 3, 4, 5\}$.

Corollary 1 *The function $\mathsf{E}_i(\mathsf{D}_i) : U^2 \to U$ in Eq. (10)a (Eq. (10)b) is a fuzzy e-subX(N)or connective.*

In addition, the family of e-subX(N)or connectives E_i (D_i) is referred as $\mathcal{E}$ ($\mathcal{D}$).

Proposition 9 *Let $\mathcal{A} : U^n \to U$ be an OWA-operator. For all $x, y \in U$, the function $\mathcal{E}_\mathcal{A}(\mathcal{D}_\mathcal{A}) : U^2 \to U$, called $(\mathcal{A}, \mathcal{E})$-**operator** $((\mathcal{A}, \mathcal{D})$-**operator**) is given by*

$$\mathcal{E}_\mathcal{A}(x,y) = 1 - \sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(1 - x - y - 2xy) \quad \left(\mathcal{D}_\mathcal{A}(x,y) = \sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(1 - x - y - 2xy)\right). (11)$$

Proof: For all $x, y \in U$. it holds that:

$$\begin{aligned}
\mathcal{E}_\mathcal{A}(x,y) &= \mathcal{A}(E_1(x,y), \ldots, (E_n(x,y)) = w_1 E_{\sigma(1)}(x,y) + \ldots + w_n E_{\sigma(n)}(x,y) \\
&= \sum_{i=1}^{n} w_i (1 - \frac{1}{\sigma(i)}(1 - x - y + 2xy)) = \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} (\frac{w_i}{\sigma(i)}(1 - x - y + 2xy)) \\
&= 1 - \sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(1 - x - y - 2xy).
\end{aligned}$$

Therefore Eq.(13) is verified. The other one can be analogously done. ▲

Proposition 10 *Let $\mathcal{A} : U^n \to U$ be an OWA-operator. For all $x, y \in U$, the $(\mathcal{A}, \mathcal{E})$-operator $((\mathcal{A}, \mathcal{D})$-operator) verifies* **Ek (Dk)** *for* $\mathbf{k} \in \{0(i), 1, 2(i), 2(ii), 3, 4\}$.

Proof: For all $x, y \in U$ and $i \geq 1$ the following holds.

E0(i) Since $\mathcal{E}_\mathcal{A}(1,0) = 1 - \sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(1 - 1 - 0 - 2 \cdot 0 \cdot 1)$ then $\mathcal{E}_\mathcal{A}$ verifies $E0(i)$.

E1 By Eq. (10), the following holds: $\mathcal{E}_\mathcal{A}(x,y) = 1 - \sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(1 - x - y - 2xy) = 1 - \sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(1 - y - x - 2yx) = 1 - \sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(1 - x - y - 2xy)$. Therefore, $\mathcal{E}_\mathcal{A}$ verifies $E1$.

E2(i) By Prop.2, if $y_1 \leq y_2$ then $\sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(1 - y_1) \geq \sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(1 - y_2)$ which means that $1 - \sum_{i=1}^{n} \frac{1}{i}(1 - y_1) \leq \sum_{i=1}^{n} 1 - \frac{1}{i}(1 - y_2)$. Therefore, $\mathcal{E}_\mathcal{A}$ verifies $E2(i)$.

E2(ii) Analogous to **E2(i)**.

E3: Since $1 - 2x + x^2 = 0$ does not have rational zeros on U, we conclude that $\mathsf{E}(x,x) \neq 1$.

E4: When $\mathcal{E}_{\mathcal{A}}(x,y) = 1$ we have that $\sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(1 - x - y - 2xy) = 0$. Then, it implies that $1 - x - y - 2xy = 0$ or $x = \frac{1-y}{1-2y}$ Thus, we have as the possible solutions ($x = 0$ and $y = 1$) or ($x = 1$ and $y = 0$).

Analogously, it can be proved to the dual construction. Therefore Prop.10 is verified. ▲

Corollary 2 *Let $\mathcal{A} : U^n \to U$ be an OWA-operator. For all $x, y \in U$, the $(\mathcal{A}, \mathcal{E})$-operator $((\mathcal{A}, \mathcal{D})$-operator) is e-subX(N)or connective.*

Proof: Straightforward from Prop.10. ▲

4.3 Aggregating Fuzzy e-X(N)or (co)implications

In the following, focusing on the fuzzy e-XNor sub(co)implication class, a fuzzy sub(co)implication is obtained from the OWA-operator and a finite subset of fuzzy e-subX(N)or sub(co)implications.

Proposition 11 *The function $I_i(\mathsf{D}_i) : U^2 \to U$ given by:*

$$I_i(x,y) = 1 - \frac{1}{i}(x - xy + 2x^2 y) \left(J_i(x,y) = \frac{1}{i}(1-x)(2 - y - 2(1-x)(1-y)) \right), \forall x, y \in U, \text{ (12)}$$

whenever i is a rational index such that $i \geq 1$, verifies $\mathbf{Ik}$ ($\mathbf{Jk}$) for $\mathbf{k} \in \{0(i), 1, 2(i), 2(ii), 3, 4, 5\}$.

Corollary 3 *The function $I_i(J_i) : U^2 \to U$ in Eq. (12)a (Eq. (12)b) is a fuzzy e-subX(N)or (co)implication.*

In addition, the family of e-subX(N)or connectives I_i (J_i) is referred as $\mathcal{I}$ ($\mathcal{J}$).

Proposition 12 *Let $\mathcal{A} : U^n \to U$ be the OWA-operator. The function $\mathcal{E}_{\mathcal{A}}(\mathcal{D}_{\mathcal{A}}) : U^2 \to U$, called $(\mathcal{A}, \mathcal{E})$-operator $((\mathcal{A}, \mathcal{D})$-operator) is given by*

$$\mathcal{I}_{\mathcal{A}}(x,y) = 1 - \sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(x + xy - 2x^2 y) \left(\mathcal{J}_{\mathcal{A}}(x,y) = \sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(1-x)(2 - y - 2(1-x)(1-y)) \right). \text{ (13)}$$

Proof: For all $x, y \in U$. it holds that:

$$
\begin{aligned}
\mathcal{I}_{\mathcal{A}}(x,y) &= \mathcal{A}(I_1(x,y), \ldots, (I_n(x,y)) = w_1 I_{\sigma(1)}(x,y) + \ldots + w_n I_{\sigma(n)}(x,y) \\
&= \sum_{i=1}^{n} w_i \left(1 - \frac{1}{\sigma(i)}(x - xy + 2x^2 y)\right) = \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} \left(\frac{w_i}{\sigma(i)}(x - xy + 2x^2 y)\right) \\
&= 1 - \sum_{i=1}^{n} \frac{w_i}{\sigma(i)}(x - xy + 2x^2 y).
\end{aligned}
$$

Therefore Eq.(13) is verified. The other one can be analogously done. ▲

Proposition 13 *Let $\mathcal{A} : U^n \to U$ be an OWA-operator. For all $x, y \in U$, the $(\mathcal{A}, \mathcal{I})$-operator $((\mathcal{A}, \mathcal{J})$-operator) verifies $\mathbf{Ek}$ ($\mathbf{Dk}$) for $\mathbf{k} \in \{0(i), 1, 2(i), 2(ii), 3, 4\}$.*

Proof: Analogous of Prop.10. ▲

Corollary 4 *Let $\mathcal{A} : U^n \to U$ be an OWA-operator. For all $x, y \in U$, the $(\mathcal{A}, \mathcal{E})$-operator $((\mathcal{A}, \mathcal{D})$-operator) is e-subX(N)or connective.*

Proof: It follows from Prop.13. ▲

5   Conclusion

The study of aggregating fuzzy e-X(N)or connectives, their dual construction and corresponding fuzzy implications is main contribution of this work. Taking the class of strong fuzzy negation (standard negation), the probablistic sum and the product t-norm the paper formally states that the OWA-operator preserves the fuzzy e-X(n)or connectives and corresponding fuzzy e-X(N)or (co)implications.

Based on the previous study of fundamental properties of other fuzzy X(N)or classes (see, e.g. [2, 7]), our current investigation aims clearly to contemplate other aggregation fuzzy X(N)or connectives also including their conjugate operators obtained by action of an automorphisms on U.

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