Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 11, n. 1, 2025.

Trabalho apresentado no XLIII CNMAC, Centro de Convenções do Armação Resort - Porto de Galinhas - PE, 2024

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

A Numerical Investigation on Iterative Methods for the Two-Dimensional Poisson Equation Discretized with High-Order Mimetic Operators

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Mimetic operators are of increasing interest to the scientific computing community due to their ability to preserve many important properties of the continuous problem (e.g., conservation laws) while maintaining the same order of accuracy on the boundary as in the interior points [2]. This mathematical framework results in locally dense and potentially ill-conditioned linear systems that are challenging to solve. This issue can partially be addressed using adequate iterative solvers, which is the focus of this work. Using the two-dimensional Poisson equation and mimetic operators of order $k \in \{2, 4\}$, we compare the computational times and number of iterations obtained with different Krylov-subspace-based iterative methods used for the resolution of the linear systems.

The numerical experiments, a follow-up from the previous work at [3], were run on a computer with Intel Core i5-12450 with 16GB RAM, using the software MATLAB R2022a for Windows 11. The chosen benchmark problem is the 2D minimal Poisson equation, subject to Robin boundary conditions:

$$-\Delta u = f \text{ in } \Omega \tag{1}$$

$$\alpha u + \beta \nabla u \bullet n = \gamma \text{ on } \Gamma, \tag{2}$$

extracted from the Mimetic Operators Library Enhanced (MOLE) [1], which provides matricial discrete analogs of the most common vector calculus operators such as gradient, divergence, and laplacian among others, to solve partial differential equations. The discretization of such PDEs results in large sparse linear systems. To solve it, the following iterative methods were employed: the restarted GMRES(m), in which case the built-in MATLAB implementation is executed; later on, LGMRES(m, k), GMRES-E(m, k), PD-GMRES(m) from the Krylov Subspace-Based Adaptive Solvers (KrySBAS) library [4]. A Jacobi preconditioner is used first to reduce the condition number of the matrices.

Numerical experiments for different discretization sizes M (of an $(M + 2)$-by-$(M + 2)$ grid), precision orders of the mimetic operators (k) and iterative methods for the resulting linear systens are compared. According to the preliminary results, shown in Table 2, a strategy for modifying the size of the search subspace adaptively is good enough to accelerate convergence up to a certain number of discretization size: in the cases when the 1D step size is smaller, even the adjustment of the restart parameter m from GMRES is not sufficient to reach convergence under the iteration constraints. Hence, an enrichment of the search subspace is of substantial help. For the selected problems, the LGMRES(m, k) has performed better in the majority of the cases.

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Table 1: List of test problems in the Poisson problem with different precision orders (k) and discretization sizes (M).

k	M	size(A)	nnz(A)	%nnz(A)	condPostJacobi(A)
2	32	1 156	5 252	0.393	603.0954
2	64	4 356	20 740	0.11	2.413E+03
2	128	16 900	82 436	0.029	9.655E+03
2	256	66 564	328 708	0.007	3.863E+04
2	512	264 196	1 312 772	0.002	1.545E+05
4	32	1 156	13 316	0.9965	843.0009
4	64	4 536	53 252	0.2806	3.3748E+03
4	128	16 900	212 996	0.0746	1.3502E+04
4	256	66 564	851 972	0.0192	5.4011E+04
4	512	264 196	3 407 876	0.0049	2.1605E+05

Table 2: Metrics for problems and iterative methods: average computational time in seconds and number of iterations required for $\|r_j\|/\|r_0\| < 10^{-9}$. For the cases where does not reach convergence before 100 restarts the method is stopped and the time is denoted as NC.

	GMRES(m)	PD-GMRES(m_j)	LGMRES	GMRES-E
k, M	Time (Iterations)	Time (Iterations)	Time (Iterations)	Time (Iterations)
2, 32	0.0202 (170)	0.051 (183))	0.0618 (150)	0.1434 (120)
2, 64	0.1491 (561)	0.1369 (391)	0.1370 (270)	0.1498 (240)
2, 128	1.9149 (1665)	0.5187 (710)	0.4962 (510)	0.5649 (540)
2, 256	NC	3.0038 (1536)	2.4703 (960)	4.2089 (1440)
2, 512	NC	NC	46.488 (1740)	NC
4, 32	0.0078 (253)	0.013 (250)	0.011 (210)	0.016 (150)
4, 64	0.233 (700)	0.138 (471)	0.126 (330)	0.125 (300)
4, 128	2.698 (2154)	0.673 (769)	0.639 (570)	0.806 (660)
4, 256	NC	4.593 (1787)	3.643 (1140)	6.638 (1770)
4, 512	NC	NC	61.96 (2010)	NC

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