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Parameter Error Analysis for Some α-Models of Turbulence

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The difficulty of dealing with 3D Navier-Stokes equations (NSE) is due to the fact that global regularity results for this model are still unknown, being one of the most challenging problems in partial differential equations theory, since the control of vorticity stretching term in the 3D vorticity equation is the main obstacle. As an analytical and computational alternative, some subgrid-scale models of turbulence, called in literature as *α-models*, were developed and have been extensively studied as a regularization of NSE, which make use of a special smoothing kernel, the one associated with the Green function of the Helmholtz operator $v = u - \alpha^2\Delta u$, where α is a given length-scale parameter. The first member of the family was introduced in the late 1990s (see [3]) called the Navier-Stokes-α (NS-α), as a closure model for the Reynolds averaged equations of the NSE. Posteriorly, similar models such as Leray-α [4] and the Modified Bardina model (see [5]) were introduced and have been object of interest specially in simulation in computational fluid dynamics. Computationally, explicit solutions for these models give excellent agreement with experimental data for a wide range of huge Reynolds numbers.

In this work, we are interested in studying $\alpha-$models in the context that the lengthscale parameter $\alpha > 0$ is not exactly known, analyzing the dynamic of the system when the parameter α is replaced by a "guess" β and estimating the error of the real state solution u when such a replacement on parameters is done. For that, suppose we have the physical system $\frac{du}{dt} = F_\lambda(u)$, with the initialization point u_0 and the parameter λ both missing. We want to construct an algorithm for approximate $u(t)$ from the available observational measurements $I_h(u(t))$. We consider then the auxiliary assimilated system

$$\frac{dw}{dt} = F_{\tilde{\lambda}}(w) - \eta I_h(w) + \eta I_h(u)$$

with $w(0) = w_0$, where w_0 is taken to be arbitrary, $\tilde{\lambda}$ is an a priori guessing to λ. For the sake of understanding, we choose Bardina model

$$\begin{aligned} v_t - \nu\Delta v + (u\cdot\nabla)u = -\nabla p + f, \quad \text{in } \Omega = [0,L]^3, \\ v = u - \alpha^2\Delta u \ \text{ with } \ \nabla\cdot v = \nabla\cdot u = 0. \end{aligned} \tag{1}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the spatial (filtered) velocity field, $p = p(x,t)$ is a modified scalar pressure field, $f = f(x,t)$ is a given external force and $\nu > 0$ is kinematic viscosity. Now, consider the continuous data assimilation technique employed firstly in [1], an algorithm that uses some types of measurement data of the system, for which a general type of approximation

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interpolation operator exists. More precisely, consider the following system

$$\begin{aligned} z_t - \nu\Delta z + (w\cdot\nabla)w &= -\nabla p + f - \eta(I_h w - I_h u), \quad \text{in } \Omega = [0,L]^3, \\ z &= w - \beta^2\Delta w \quad \text{with } \nabla\cdot v = \nabla\cdot u = 0, \end{aligned} \tag{2}$$

where $\eta > 0$ is a relaxation parameter and I_h is an interpolant linear operator that can be constructed from observational measurements of the system (1), with $h > 0$ the coarse spatial resolution related to the accuracy of the operator, satisfying for all $\phi \in H^2(\Omega)$,

$$\|I_h\phi - \phi\|^2_{L^2(\Omega)} \leq c_1 h^2 \|\nabla\phi\|^2_{L^2(\Omega)} + c_2 h^4 \|\Delta\phi\|^2_{L^2(\Omega)}. \tag{3}$$

This work is inspired in [2], where the authors studied the same continuous data assimilation algorithm in the context of an unknown viscosity for 2D NSE. In our work, we obtained the following summary result, related to the convergence of the approximate solution to the real state solution:

Theorem 1. *Let $f \in L^\infty(\mathbb{R}_+, L^2)$ and u and w solutions of (1) and (2), respectively, with I_h satisfying (3). Choosing $\eta > 0$ large enough and $h > 0$ small enough, we have for all $t \geq 0$,*

$$\begin{aligned} \|w(t) - u(t)\|^2_{L^2(\Omega)} &+ \beta^2\|\nabla w(t) - \nabla u(t)\|^2_{L^2(\Omega)} \\ &\leq e^{-\frac{\lambda_1\nu}{2}t}(\|w(0) - u(0)\|^2_{L^2(\Omega)} + \beta^2\|\nabla w(0) - \nabla u(0)\|^2_{L^2(\Omega)}) \\ &+ \frac{|\beta^2 - \alpha^2|^2}{\beta^2}\cdot C(\alpha, \nu, \|f\|_{L^2(\Omega)}, \lambda_1, M) \end{aligned}$$

where λ_1 is the first eigenvalue of Laplacian operator and M is an upper bound for the L^2-norm of the real state solution u and its gradient ∇u.

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