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Realization of Bipartite Weighted Graphs by Stable Gauss Maps

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The singularities of a stable Gauss map of a closed orientable surface immersed generically in three-dimensional Euclidean space, according to H. Whitney's Theorem, are of the fold and cusp types. The singular set of a stable Gauss map of a surface, which consists of curves of fold points containing isolated cusp points, is the parabolic set on the surface. Each parabolic curve of the singular set separates a hyperbolic region from an elliptic region of the surface. In this work, we will explore how weighted graphs can be associated with stable Gauss maps and present a general result that determines necessary and sufficient conditions for a weighted graph to be associated with a stable Gauss map.

Let M be a closed orientable surface immersed $f : M \to \mathbb{R}^3$, with stable Gauss map $\mathcal{N}_f : M \to S^2$, then we can define three subsets on M based on their Gaussian curvature $K(p) = \det d_p(\mathcal{N}_f)$, where $p \in M$:

- the **eliptic region** is the set $\{p \in M; K(p) > 0\}$;
- the **parabolic region** is the set $\{p \in M; K(p) = 0\}$;
- the **hyperbolic region** is the set $\{p \in M; K(p) < 0\}$;

We then associate a connected bipartite weighted graph $\mathcal{G}$ to the surface M through its Gaussian curvature:

1) each elliptic and hyperbolic region corresponds to a vertex. We label the vertices with a positive sign $(+)$ for elliptic regions and with a negative sign $(-)$ for hyperbolic regions;

2) each parabolic curve corresponds to an edge connecting the vertices of two bordering regions (one elliptic and the other hyperbolic, since the surface is orientable);

3) the genus of each elliptic or hyperbolic region corresponds to the weight of its associated vertex;

In such case, we say that the weighted bipartite graph $\mathcal{G}$ is *realized* by a stable Gauss map of a closed orientable surface M (or equivalently, the stable Gauss map of a closed orientable surface M *realizes* $\mathcal{G}$) if and only if there exists an immersion $f : M \to \mathbb{R}^3$ whose stable Gauss map $\mathcal{N}_f$ has $\mathcal{G}$ as its associated graph.

Our goal is to prove that given any bipartite weigthed graph $\mathcal{G}$, we can find a closed orientable surface M with associated stable Gauss map which realizes $\mathcal{G}$ such that $\chi(M) = 2(1-\beta_1(\mathcal{G})+\omega(\mathcal{G}))$, where $\beta_1(\mathcal{G})$ is the number of independent cycles of the graph $\mathcal{G}$ and $\omega(\mathcal{G})$ is the sum of all weights of the vertices of $\mathcal{G}$. For that, we first prove two partial results:

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Theorem 1: Any tree (connected graph with no cycles) with zero weight can be realized by a stable Gauss map on an embedded sphere.

Theorem 2: Any connected bipartite graph with zero weight can be realized by a stable Gauss map of a closed orientable surface M, with $\chi(M) = 2(1 - \beta_1(\mathcal{G}))$.

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References

[1] T. Banchoff, T. Gaffney, C. Mccrory, and D. Dreibelbis. **Cusps of Gauss Mappings**. London: Pitman Books Limited, 1982. ISBN: 9780273085362. URL: https://www.emis.de/monographs/CGM/index.html.

[2] M. P. do Carmo. **Geometria diferencial de curvas e superfícies**. 5ª ed. Textos Universitários. Rio de Janeiro: SBM, 2012. ISBN: 9788585818265.

[3] C. M. de Jesus, S. M. Moraes, and M. C. Fuster. "Stable Gauss maps from a global viewpoint". In: **Bulletin of the Brazilian Mathematical Society** 42 (2012), pp. 87–103. DOI: 10.1007/s00574-011-0005-8.

[4] C. M. de Jesus and E. Sanabria-Codesal. "Realization of graphs by fold Gauss maps". In: **Topology and its Applications** 234 (2018). DOI: 10.1016/j.topol.2017.11.016.

[5] C. P. Mesquita. **Grafos e suas aplicações no Ensino da Matemática**. 2021.

[6] I. V. Souza. "Singularidades de Aplicações de Gauss Estáveis". Master dissertation. UFV, 2012.

[7] H. Whitney. "On singularities of mappings of euclidean spaces i: Mappings of the plane into the plane". In: **Annals of Mathematics** 62 (1955), pp. 374–410. DOI: 10.2307/1970070.