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ANN-Flux Method to the Solution of Nonlinear Conservation Laws: the Shock and the Rarefaction Cases

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Artificial neural networks (ANNs, [3]) have been successfully applied to solve partial differential equations, mainly after the emergence of the physics-informed neural networks (PINNs, [6]). Applications to nonlinear conservation laws are also notable, including PINNs for high-speed flows [5], conservative PINNs (cPINNs, [4]), and weak PINNs (wPINNs, [7]). In this work, we propose the new ANN-Flux method to solve the Riemann problem for the nonlinear conservation law

$$u_t + (F(u))_x = 0,\ x, t \in \mathbb{R} \times (0, t_f], \tag{1}$$

$$u(x,0) = u_L,\ x \in \mathbb{R}_-,\ u(x,0) = u_R,\ x \in \mathbb{R}_+, \tag{2}$$

with given left and right states $u_L, u_R \in \mathbb{R}$, and flux function $F : \mathbb{R} \to \mathbb{R}$. It is well-known that the entropic solution of (1) with a convex flux function is either a shock or a rarefaction wave depending on the left and right states. The exact solution of the Riemann problem is given by (left: shock, $u_R < u_L$, right: rarefaction, $u_L < u_R$)

$$u(x,t) = \begin{cases} u_L & , x < t\sigma, \\ u_R & , x > t\sigma, \end{cases} \quad u(x,t) = \begin{cases} u_L & , x < tF'(u_L), \\ G(x/t) & , tF'(u_L) < x < tF'(u_R), \\ u_R & , x > tF'(u_R), \end{cases} \tag{3}$$

where $G(u) = [F']^{-1}(u)$ and σ is the shock speed satisfying the Rankine-Hugoniot condition $\sigma = \frac{F(u_L) - F(u_R)}{u_L - u_R}$. The ANN-Flux (ANN-F) method is designed for complex fluxes, which are costly to be evaluated, differentiated or inverted. Once multilayer perceptrons (MLP, [3]) neural networks are universal approximators [3], ANN-F consists of approximating the flux F by a MLP $\tilde{y} = \mathcal{N}_F(u)$ classically trained to minimize the mean squared error (MSE) loss $\varepsilon\left(\tilde{y}^{(s)}, y^{(s)}\right)$ with a generated data set $\left\{\left(u^{(s)}, y^{(s)} = F\left(u^{(s)}\right)\right)\right\}_{s=1}^{n_{s,F}}$, where $n_{s,F}$ is a given number of samples. For the simple shock wave case, the solution is then approximated by substituting F by $\mathcal{N}_F$ in (3)(left). But, for a rarefaction case, a second MLP neural network $\mathcal{N}_G$ learns to approximate $[\mathcal{N}_F']^{-1} \approx [F']^{-1}$. The evaluation of $\mathcal{N}_F'$ can be efficiently computed by automatic differentiation (AD, [2]). The $\mathcal{N}_G$ is trained using a directly generated data set $\left\{\left(\delta\tilde{y}^{(s)}, u^{(s)}\right)\right\}_{s=1}^{n_{s,G}}$, where $n_{s,G}$ is a given number of samples. The data $\left\{u_L \leq u^{(s)} \leq u_R\right\}_{s=1}^{n_{s,G}}$ is randomly generated, forward through N_F to give $\tilde{y}^{(s)} = N_F\left(u^{(s)}\right)$, and then backward propagated to compute $\delta\tilde{y}^{(s)} = \mathcal{N}_F'\left(u^{(s)}\right)$. The training of $\tilde{u}^{(s)} = \mathcal{N}_G\left(\delta\tilde{y}^{(s)}\right)$ is obtained by minimizing the MSE loss $\varepsilon\left(\tilde{u}^{(s)}, u^{(s)}\right)$ on the neural network weights and biases. The solution of the rarefaction case is then given by substituting F' by $\mathcal{N}_F'$ and G by $\mathcal{N}_G$ in (3)(right).

As a preliminary result, Figure 1 shows the comparison between ANN-F and analytical solutions of the Burgers equation, i.e. equation (1) with the flux $F(u) = u^2/2$. By following a trial and error

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strategy, the $\mathcal{N}_F$ has been chosen as a MLP of $1 - 30 \times 3 - 1$ architecture (one input, 3 hidden layers each of 30 neuron units, one output) with hyperbolic tangent and the identity as activation functions at hidden and output layers, respectively. The $\mathcal{N}_G$ has been set as a simple perceptron with the identity as the activation function. Both networks have been trained with the Adam optimizer, the learning rate $l_r = 10^{-3}$, and the tolerance of $\tau = 10^{-5}$ for the loss function. We note that the ANN-F has obtained very good results of the Burgers equations. Further work will include its application to more complex conservation laws, as for instance the traffic flow problem with a nonlocal flux based on the Lighthill-Witham-Richards model for vehicular traffic [1].

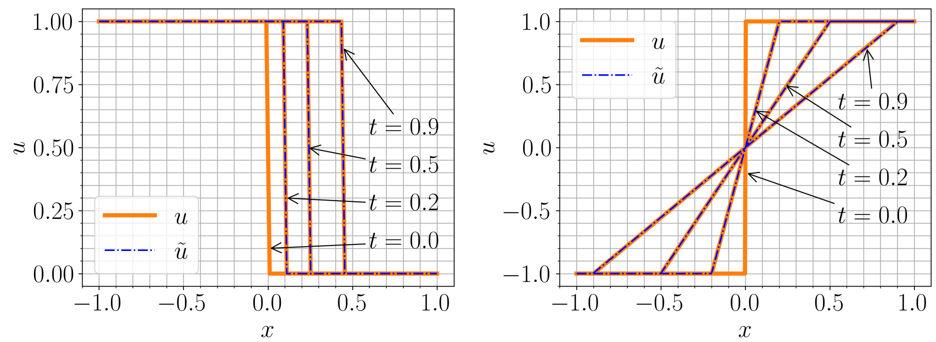

Figure 1: ANN-F ($\tilde{u}$) *versus* analytical solutions (u) of Burgers equation. Left: shock. Right: rarefaction. Source: Authors

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