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Recovering Isometry Groups from Killing Vector Fields

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On pure geometric grounds, Killing vector fields [8] play a central role in Riemanian manifolds. They serve as elements of tangent planes, generating local isometries and, in some approaches, could be used to construct global conserved structures on general relativistic spacetimes [4]. Its arms extends to different areas, ranging from classical mechanics [9] to the description of constant-curvature spaces of a homogeneous isotropic and static spacetimes [3, 6] passing through, recently, to their meaning in comprehending fluid flows on curved surfaces [12]. According to [11], there are several approaches to study these vector fields on Lorentzian manifolds (i.e., pseudo-Riemannian manifolds which admit a pseudo-metric η). One of them is to employ proper actions of Lie groups. However, it's not explicitly addressed how to recover the Lie group using the expression of Killing vector fields, which is the case in [7] as well.

On the other side, Lie groups [5] act on the manifold, inducing a linear representation on its tangent space, representing metric-preserving global transformations. There are interesting results relating these two approaches. For example, it is known that the set of Killing fields on $\mathcal{M}$ is indeed a Lie algebra [1]. Even more, for the case of curved spaces, if the Riemann curvature tensor vanishes at some point $p \in \mathcal{M}$, then the Lie algebra of Killing vector fields is related to a sub-algebra of $SO(r, q)$, where the pair (r, q) denotes the signature of the local pseudo-metric of the tangent space at p [2]. Hence, leveraging on the previous cited results, our main point of investigation lies on the study of, up to our knowledge, a not well explored connection between these two topics.

More specifically, our investigation takes a significant turn as we identify a connection between Killing vector fields of $(\mathbb{R}^n, \delta)$ and $(\mathbb{R}^{1+n}, \eta)$ (for $n \geq 2$) and Lie algebra elements by interpreting them as induced vector fields, with the definition of the latter being presented in [8]. Furthermore, the connection we demonstrate here between this definition and Killing vector fields as generators of Lie algebras, to the best of our knowledge, has not been thoroughly investigated in the literature. We delve into specific examples of our construction, namely, $(\mathbb{R}^3, \delta)$ and $(\mathbb{R}^4, \eta)$. This, in turn, when identifying the Killing vector fields of these manifolds as induced vector fields, allows us to reconstruct the Lie groups that preserve the respective metrics, which are, in the first instance, the special Euclidean group $SE(3)$, whereas the Poincaré group emerges in the second case.

Our findings contribute to a deeper understanding of the interplay between geometry and group theory, reinforcing the bridge between the intrinsic geometry of manifolds and the algebraic structures governing their symmetries. Thus, we summarize our main result in the following theorem.

Theorem 1. *Let $(\mathbb{R}^n, \delta)$ and $(\mathbb{R}^{1+n}, \eta)$ represent the $n-$dimensional Euclidean space and the $1 + n-$dimensional Minkowski spacetime ($n \geq 2$), respectively. Suppose we define their Killing vector fields as induced vector fields by Lie algebra elements. Then, they generate the corresponding isometry groups: the special Euclidean group $SE(n)$ in the former and the Poincaré group in the latter.*

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The proof of the Theorem 1, as well as an extended version of this work, can be found in [10].

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