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Dynamic Programming Principle and Hamilton-Jacobi Equation for Optimal Control Problems with Uncertainty

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This work establishes that the value function for Mayer's problem in a Hilbert space is the unique lower semi-continuous solution to the Hamilton-Jacobi-Bellman (HJB) equation under specific conditions. By investigating a parametrized Riemann–Stieltjes problem, we achieve the compactness of its trajectories, which, combined with a characterization of the lower semicontinuity of the associated value function, establishes the existence of optimal controls. Subsequently, utilizing the differential inclusion approach and prior results, we prove the uniqueness of the solution to the HJB equation

This work aims to prove that the value function for Mayer's problem, defined in a Hilbert space, is the unique lower semi-continuous solution of the Hamilton-Jacobi-Bellman equation when the nonlinear dynamics are measurable in time and the cost is an integral functional. Specifically, we investigate the parametrized Riemann–Stieltjes problem denoted by $(P)_{s,\varphi}$, concerning the initial time s and the initial states, which are ω-dependent and represented by the mapping $\varphi \in L^2(\mu, \Omega; \mathbb{R}^n)$:

$$\min \int_\Omega g(x(T,\omega),\omega)d\mu(\omega),$$

s.t.

$$\begin{cases} \dot{x}(t,\cdot) = f(t, x(t,\cdot), u(t), \cdot), \quad \text{a.e. } t \in [s,T], \\ x(s,\cdot) = \varphi(\cdot), \\ u(t) \in \mathbf{U}, \quad \text{a.e. } t \in [s,T], \quad \mathbf{U} \subset \mathbb{R}^m \quad \text{compact}, \end{cases} \tag{1}$$

for every $\omega \in \Omega$ and $s \leq t \leq T$, where (Ω, d_Ω, μ) is a compact metric measure space, with control $u \in L^\infty(0,T;\mathbf{U})$.

Initially, following the approach used in [2, 3] for the semilinear case, we establish, under certain hypotheses on the dynamics (referred to as $\mathbf{H}$ and $\mathbf{C}$) and on the measure μ (referred to as $\mathbf{H}_\mu$), that the set of trajectories defined by

$$S_{[s,T]}(\varphi) := \left\{ x \in C([s,T] : L^2(\mu,\Omega;\mathbb{R}^n)) : \ x \text{ solves (1) and } \ x(s,\cdot) = \varphi(\cdot) \right\}$$

is compact in an appropriate space of functions. Subsequently, we provide a characterization of the lower semicontinuity of the associated value function for the problem $(P)_{s,\varphi}$, defined by

$$V(s,\varphi) = \inf \left\{ \int_\Omega g(x(T,\omega),\omega)d\mu(\omega) : x \in S_{[s,T]}(\varphi) \right\}$$

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which establishes the existence of optimal trajectories.

The existence of minimizers, combined with invariance principles and the Dynamic Programming Principle, will pave the way to prove that the value function defined above is the unique lower semicontinuous solution of the following Hamilton-Jacobi-Bellman equation, defined in an infinite-dimensional space.

$$\begin{cases} -\left[V_t(t,\varphi) + H(t,\varphi,V_\varphi(t,\varphi))\right] = 0, \\ V(T,\varphi) = \displaystyle\int_\Omega g\left(\varphi(\omega),\omega\right) d\mu(\omega), \end{cases}$$

where $H : [0,T] \times L^2(\mu,\Omega;\mathbb{R}^n) \times L^2(\mu,\Omega;\mathbb{R}^n)^* \to \mathbb{R}$ is the Hamiltonian function given by

$$H(t,\varphi,p) := \inf_{u(t)\in\mathbf{U}} \left\langle p, f(t,\varphi(\cdot),u(t),\cdot)\right\rangle.$$

The proof strategy employs the differential inclusion approach and utilizes some results from [1]. Specifically, we define the set-valued map

$$F : [0,T] \times L^2(\mu,\Omega;\mathbb{R}^n) \rightsquigarrow L^2(\mu,\Omega;\mathbb{R}^n)$$

given by $F(t,\varphi) = f(t,\varphi,U(t),\cdot)$. The associated differential inclusion is then expressed as:

$$\dot{x}(t,\cdot) \in F(t,x(t,\cdot)) \quad \text{a.e. } t\in[s,T] \quad \text{with} \quad x(s,\omega) = \varphi(\omega). \;\; \forall\,\omega\in\Omega.$$

Finally, we prove the principal result of this work:

Theorem 0.1. *Let us assume that* $(\mathbf{H})$, $(\mathbf{H}_\mu)$ *and* ***(C)*** *hold true. Then the value function* V *of problem* $(P)_{s,\varphi}$ *is the unique lower semi-continuous, bounded below function such that there exists a set* $I \subset [0,T)$ *of full measure for which, for every* $(t,\varphi,\alpha) \in \mathrm{epi}V \cap \left(I \times L^2(\mu,\Omega;\mathbb{R}^n)\times\mathbb{R}\right)$*, one has*

$$\xi_0 + \min_{v\in F(t,\varphi)} \langle v,\xi\rangle = 0 \qquad \forall\,(\xi_0,\xi,-q) \in N^P_{\mathrm{epi}V}(t,\varphi,\alpha),$$

$$V(T,\varphi) = \int_\Omega g\left(\varphi(\omega),\omega\right) d\mu(\omega).$$

Where $(\xi_0,\xi) \in \partial_P V(t,\varphi)$*, the proximal subdifferential, or* P*-subdifferential of* V*.*

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