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A Simple Proof for a Particular Case of Szabados's Conjecture

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Let $\mathcal{A}$ be an alphabet with at least two elements. The elements of $\mathcal{A}^{\mathbb{Z}^2}$, called **configurations**, have the form $\eta = (\eta_{\boldsymbol{g}})_{\boldsymbol{g}\in\mathbb{Z}^2}$, where $\eta_{\boldsymbol{g}} \in \mathcal{A}$ for all $\boldsymbol{g} \in \mathbb{Z}^2$. For each $\boldsymbol{u} \in \mathbb{Z}^2$, the **shift application** $T^{\boldsymbol{u}} : \mathcal{A}^{\mathbb{Z}^2} \longrightarrow \mathcal{A}^{\mathbb{Z}^2}$ is defined by $(T^{\boldsymbol{u}}\eta)_{\boldsymbol{g}} = \eta_{\boldsymbol{g}-\boldsymbol{u}}$ for all $\boldsymbol{g} \in \mathbb{Z}^2$ and all $\eta \in \mathcal{A}^{\mathbb{Z}^2}$. A configuration $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ is said to be **periodic** if there exists a non-zero vector $\boldsymbol{h} \in \mathbb{Z}^2$, called **period** of η, such that $T^{\boldsymbol{h}}\eta = \eta$. If η has two periods linearly independents over $\mathbb{R}^2$, we say that η is **fully periodic**.

From now on, we will assume that $\mathcal{A}$ is a finite alphabet.

Let $Orb\,(\eta) = \{T^{\boldsymbol{u}}\eta : \boldsymbol{u} \in \mathbb{Z}^2\}$ denotes the $\mathbb{Z}^2$**-orbit of** $\boldsymbol{\eta} \in \boldsymbol{\mathcal{A}}^{\mathbb{Z}^2}$. If A is endowed with the discrete topology, then $\mathcal{A}^{\mathbb{Z}^2}$ equipped with the product topology is a metrizable compact space. In particular, for all $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, the set $\overline{Orb\,(\eta)}$, where the bar denotes the closure, is a compact **subshift**, i.e., a closed subset invariant by the $\mathbb{Z}^2$-action $T^{\boldsymbol{u}}$, $\boldsymbol{u} \in \mathbb{Z}^2$. We say that a configuration $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ has **low pattern complexity** if $|\{(T^{\boldsymbol{u}}\eta)|_{\mathcal{S}} : \boldsymbol{u} \in \mathbb{Z}^2\}| \leq |\mathcal{S}|$ holds for some non-empty, finite set $\mathcal{S} \subset \mathbb{Z}^2$, where $\cdot|_{\mathcal{S}}$ means the restriction to the set $\mathcal{S}$. If in addition $\mathcal{S}$ is **convex**, i.e., a subset of $\mathbb{Z}^2$ whose convex hull in $\mathbb{R}^2$, denoted by $\mathrm{Conv}(\mathcal{S})$, is closed and $\mathcal{S} = \mathrm{Conv}(\mathcal{S}) \cap \mathbb{Z}^2$, we say that η has **low convex pattern complexity**.

Employing results from algebraic geometry, Jarkko Kari and Michal Szabados [1, 2] proved the following periodic decomposition theorem:

Theorem 1.1 (Kari and Szabados [1])**.** *Let $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, with $\mathcal{A} \subset \mathbb{Z}$, be a low pattern complexity configuration. Then there exist periodic configurations $\eta_1, \ldots, \eta_m \in \mathbb{Z}^{\mathbb{Z}^2}$ such that $\eta = \eta_1 + \cdots + \eta_m$.*

Let $\ell \subset \mathbb{R}^2$ be a line through the origin. Given $t > 0$, the t-neighbourhood of ℓ is defined as $\ell^t = \{\boldsymbol{g} \in \mathbb{Z}^2 : \mathrm{Dist}(\boldsymbol{g}, \ell) \leq t\}$, where Dist denotes the Euclidean distance between a point and a set. Following Boyle and Lind [3], we say that ℓ is an **expansive** line on $\overline{Orb\,(\eta)}$ if there exists $t > 0$ such that

$$\forall\ x, y \in \overline{Orb\,(\eta)}, \quad x|_{\ell^t} = y|_{\ell^t} \implies x = y.$$

Otherwise, ℓ is called a **nonexpansive** line on $\overline{Orb\,(\eta)}$. A particular case of Boyle-Lind Theorem [3, Theorem 3.7] implies that there exists at least one nonexpansive line on $\overline{Orb\,(\eta)}$ if the subshift $\overline{Orb\,(\eta)}$ is infinite. We remark that nonexpansiveness is the heart of recent advances related to Nivat's conjecture.

If $\eta = \eta_1 + \cdots + \eta_m$ is a minimal periodic decomposition, where by minimal we mean a periodic decomposition with the smallest possible number of periodic configurations, it is easy to see that every nonexpansive line on $\overline{Orb\,(\eta)}$ contains a period for some η_i, with $1 \leq i \leq m$. In his Ph.D. thesis [4], Michal Szabados conjectured that the converse also holds:

Conjecture 1.1 (Szabados)**.** *Let $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, with $\mathcal{A} \subset \mathbb{Z}$, be a not fully periodic configuration and suppose $\eta = \eta_1 + \cdots + \eta_m$ is a minimal periodic decomposition. If $\ell \subset \mathbb{R}^2$ is a line through the origin and there exists $1 \leq i \leq m$ such that ℓ contains a period for η_i, then ℓ is a nonexpansive line on $\overline{Orb\,(\eta)}$.*

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Szabados's conjecture is a very recent open problem in symbolic dynamics. In [5], the author solves Szabados's conjecture for low convex pattern configurations. Further partial results related to Szabados's conjecture are given in [6].

In this work, we present a simple proof for Szabados's conjecture in the case that $\eta = \eta_1 + \eta_2 + \eta_3$ is a minimal periodic decomposition. The main idea for the proof is the following: let p be a prime large enough so that $\mathcal{A} \subset \{0, 1, 2, \ldots, p-1\}$ and consider the periodic decomposition $\bar{\eta} = \bar{\eta}_1 + \bar{\eta}_2 + \bar{\eta}_3$, where the bar denotes the congruence modulo p. Since each configuration $\bar{\eta}_i$ is defined on a finite alphabet, it is easy to see that a line through the origin containing a period for $\bar{\eta}_i$ is nonexpansive. By the pigeonhole principle, we may extend this nonexpansiveness from $\bar{\eta}_i$ to $\bar{\eta}$. The result follows from the fact that $\bar{\eta}$ and η are essentially the same configurations.

Referências

[1] J. Kari e M. Szabados. "An Algebraic Geometric Approach to Nivat's Conjecture". Em: **Information and Computation** 271 (2020), pp. 104–481.

[2] J. Kari e M. Szabados. "An Algebraic Geometric Approach to Nivat's Conjecture". Em: **In Automata, Languages, and Programming - 42nd International Colloquium, ICALP 2015, Kyoto, Japan, Proceedings, Part II** (2015), pp. 273–285.

[3] M. Boyle e D. Lind. "Expansive Subdynamics". Em: **Trans. Amer. Math. Soc.** 349 (1997), pp. 55–102.

[4] M. Szabados. "An Algebraic Approach to Nivat's Conjecture". Tese de doutorado. University of Turku, 2018.

[5] C. F. Colle. "On Periodic Decompositions and Nonexpansive Lines". Em: **Mathematische Zeitschrift** 303.80 (2023).

[6] C. F. Colle. "On periodic decompositions, one-sided nonexpansive directions and Nivat's conjecture". Em: **Discrete and Continuous Dynamical Systems** 43.12 (2023), pp. 4299–4327.

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