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Constant rank-type constraint qualifications and second-order optimality conditions

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Abstract. The Constant Rank Constraint Qualification (CRCQ), introduced by Janin in [Math. Program. Study 21:110-126, 1984], has several applications in nonlinear programming context, such as computing the derivative of the value function, second-order optimality conditions, global convergence, stability analysis and encompass without entirely the linear program problems. This work will present an extension of CRCQ that retrieves the well known properties from nonlinear programming and, in addition, to propose a constraint qualification based on curves that naturally rises from CRCQ and explain in a very simple way the second-order optimality conditions that can be obtained for second-order cone programming problems.

Keywords. second-order cone programming, constant rank, constraint qualification, second-order optimality conditions.

1 Introduction

The study of optimality conditions and constraint qualifications has been shown to play an important role in optimization, once important works has been published in recent years, such as the sequential optimality conditions for second-order cone programming problems [7]. See also references therein.

To the best of our knowledge, the first approach to define a constant rank-type constraint qualification for second-order cone programming problems was in [14], where the authors proposed CRCQ, RCRCQ and CRSC conditions. However, in [3] we showed that their proposals had some mistakes that we would investigate later. The example proposed in [3] and an example given by [1] showed that a constant rank condition for second-order cone programming problems would take into account properly the structure of the cone, once even a problem where its constraints are linear might not have Lagrange multipliers for a local minimizer. To deal with these difficulties, diverse approaches were made in order to converge to a solution for this problem. First, we presented a naive approach in [4], where we used a reduction mapping in order to transform some conic constraints in to inequality constraints, for then to use theoretical framework from nonlinear programming. However, for the remaining sequences we had to use Robinson's CQ. It is important to notice that our approach can deal with both types of constraints at the same time. Later, instead of avoid the second-order cone structure, we embraced it and through the eigenvector structure, we could then make an other approach weakening the nondegeneracy condition and using the sequential optimality condition. See [6] for more details. Last, we used the constant rank theorem in a similar vein that Janin did in [11] for nonlinear programming problems. We could reclaim similar results as in nonlinear programming problems such as: it is weaker than nondegeneracy condition, independent of Robinson's CQ and stronger than Abadie's CQ. In addition, we could

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get some second-order results based on the critical cone such as [2] did for nonlinear programming problems. See [5].

In Section 2 we start present the basic definitions for the second-order cone programming problems. In Section 3 we present the first-order optimality conditions and the difficulties to define a constant rank-type constraint qualification. In Section 4 we present our proposal based on the constant rank theorem and its applications, such as a new constraint qualification based on curves and the second-order conditions obtained with the proposals.

2 The Second-Order Cone Programming problem

Let us consider the standard nonlinear second-order cone programming problem

$$\begin{array}{ll} \text{Minimize} & f(x), \\ \text{s.t.} & g_j(x) \in \mathbb{K}_{m_j}, \quad j = 1, \ldots, q, \end{array} \tag{NSOCP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$ are at least twice continuously differentiable and $\mathbb{K}_{m_j}$ is a second-order cone (also knonw as Lorentz cone), that is given by $\mathbb{K}_{m_j} := \{(z_0, \widehat{z}) \in \mathbb{R} \times \mathbb{R}^{m_j-1} \mid z_0 \geq \|\widehat{z}\|\}$ when $m_j > 1$ and $\mathbb{K}_1 := \{x \in \mathbb{R} \mid x \geq 0\}$. We will denote the feasible set of (NSOCP) by Ω. The (Bouligand) *tangent* cone is given by

$$\mathcal{T}_\Omega(\bar{x}) := \{d \in \mathbb{R}^n \mid \exists\, t_k \to 0^+,\ \exists\, d^k \to d \text{ such that } \bar{x} + t_k d^k \in \Omega\}. \tag{1}$$

With respect to the topological part the $\mathbb{K}_m$, the interior part of $\mathbb{K}_{m_j}$ is $\text{int}(\mathbb{K}_{m_j}) := \{(z_0, \widehat{z}) \in \mathbb{R}\times\mathbb{R}^{m_j-1} \mid z_0 > \|\widehat{z}\|\}$ and the nonzero boundary is $\text{bd}^+(\mathbb{K}_{m_j}) := \{(z_0, \widehat{z}) \in \mathbb{R}\times\mathbb{R}^{m_j-1} \mid z_0 = \|\widehat{z}\| > 0\}$. Given a feasible point $\bar{x}$, consider the following index sets: $I_B(\bar{x}) := \{j; g_j(\bar{x}) \in \text{bd}^+(\mathbb{K}_{m_j})\}$, $I_{\text{int}}(\bar{x}) := \{j; g_j(\bar{x}) \in \text{int}(\mathbb{K}_{m_j})\}$ and $I_0(\bar{x}) := \{j; g_j(\bar{x}) = 0$.

With these sets at hand, let us introduce the *linearized* cone $\mathcal{L}_\Omega(\bar{x})$:

$$\begin{aligned} \mathcal{L}_\Omega(\bar{x}) &:= \{d \in \mathbb{R}^n \mid Dg_j(\bar{x})d \in \mathcal{T}_{\mathbb{L}_{m_j}}(g_j(\bar{x})),\ j = 1, \ldots, q\} \\ &= \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} Dg_j(\bar{x})d \in \mathbb{K}_{m_j}, & j \in I_0(\bar{x}); \\ \langle Dg_j(\bar{x})d, \Gamma_j g_j(\bar{x})\rangle \geq 0, & j \in I_B(\bar{x}) \end{array} \right\}, \end{aligned} \tag{2}$$

where Γ_j is a diagonal matrix where $(\Gamma_j)_{ii} = 1$ if $i = 1$ and $(\Gamma_j)_{ii} = -1$ otherwise. The expression showed in (2) was obtained in [8, Lemma 25] and it gives us a easier way to deal with the linearized cone, that will be explored along this work.

Given a feasible point $\bar{x}$ of (NSOCP), we say that the Karush-Kuhn-Tucker (KKT) conditions hold for problem (NSOCP) at feasible point $\bar{x}$ if there exists Lagrange multipliers $\mu_j \in \mathbb{K}_{m_j}$, $j = 1, \ldots, q$ such that

$$\nabla_x L(\bar{x}, \mu) = \nabla f(\bar{x}) - \sum_{j=1}^{q} Dg_j(\bar{x})^T \mu_j = 0, \tag{3}$$

$$\langle \mu_j, g_j(\bar{x})\rangle = 0, \quad j = 1, \ldots, q, \tag{4}$$

where $L(x, \mu) := f(x) - \sum_{j=1}^{q} \langle \mu_j, g_j(x)\rangle$ is the Lagrangian function of problem (NSOCP) and $\nabla_x L(x, \mu)$ is the gradient of L at the point (x, μ) with respect to the first variable. We will denote the set of all Lagrange multipliers associated to a feasible point $\bar{x}$ by $\Lambda(\bar{x})$.

Combining the information given by the characterization of the linearized cone and the complementarity condition given in (4), we obtain that the Lagrange multiplier must has the following structure: $\mu_j = 0$ if $j \in I_{\text{int}}(\bar{x})$, $\mu_j \in \mathbb{K}_{m_j}$ if $j \in I_0(\bar{x})$ or $\mu_j = \alpha_j \Gamma_j g_j(\bar{x})$ if $j \in I_B(\bar{x})$, for some $\alpha_j \geq 0$. In other words, the constraints g_j such that $j \in I_B(\bar{x})$ seems to have a behavior similar

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to an inequality constraint from nonlinear programming problem. Indeed, with this information in mind, the KKT conditions can be rewritten in the following way: there are $\mu_j \in \mathbb{K}_{m_j}$, $j \in I_0(\bar{x})$ and $\alpha_j \geq 0$, $j \in I_B(\bar{x})$ such that

$$\nabla f(\bar{x}) - \sum_{j \in I_0(\bar{x})} Dg_j(\bar{x})^T \mu_j - \sum_{j \in I_B(\bar{x})} \alpha_j \nabla \tilde{\phi}_j(\bar{x}) = 0, \tag{5}$$

with

$$\tilde{\phi}_j(x) := \frac{1}{2}\left([g_j(x)]_0^2 - \|\widehat{g_j}(x)\|^2\right) \text{ and } \nabla \tilde{\phi}_j(x) = Dg_j(x)^T \Gamma_j g_j(x), j \in I_B(\bar{x}) \tag{6}$$

and it is called *reduction mapping.* See [8] for more details. Thus, we can then rewrite the (NSOCP) problem locally around $\bar{x}$ changing the "second-order cone-type" constraints $j \in I_B(\bar{x})$ by inequality constraints as it is done in nonlinear programming problems.

3 Optimality conditions and constraint qualification

Before we present the extension of CRCQ to (NSOCP) problem, let us understand properly the difficulties of this task. Let us start recalling the so-called *first-order geometric necessary condition.* Let $\bar{x}$ be a local minimizer of (NSOCP), then $-\nabla f(\bar{x}) \in \mathcal{T}_\Omega(\bar{x})^\circ$. This relation shows a genuine optimality condition in the sense that it is a condition that is satisfied by every local minimizers. This condition means that there is no feasible descent direction from $\bar{x}$. However, despite the fact that it is a simple condition to understand, it is hard to verify its fulfilment. In order to deal with this issue, one could ask about avoid using the polar of the tangent cone. The natural option that rises is the linearized cone. It is known that $\mathcal{T}_\Omega(\bar{x}) \subseteq \mathcal{L}_\Omega(\bar{x})$, which means $\mathcal{L}_\Omega(\bar{x})^\circ \subseteq \mathcal{T}_\Omega(\bar{x})^\circ$. Unfortunately, the polar of the linearized cone can be strictly included at the polar of tangent cone, and then we can not use the first-order geometric necessary condition. The conditions that make this gap empty are called constraint qualification.

To start the comprehension of constraint qualification in (NSOCP) problem, we could use a similar approach of nonlinear programming problems. Nevertheless, the following example given in [1, Subsection 2.1] shows that we need to consider more details in order to define a constraint qualification in (NSOCP).

Example 3.1. *Consider the following problem*

$$\begin{array}{ll} \text{Minimize} & f(x) := -x_2, \\ \text{s.t.} & g(x) := (x_1, x_1, x_2) \in \mathbb{K}_3. \end{array} \tag{7}$$

The point $\bar{x} = (0,0)$ is a local minimizer and, moreover, we have that $\mathcal{T}_\Omega(\bar{x}) = \mathcal{L}_\Omega(\bar{x})$. However, there is no Lagrange multiplier and then the KKT conditions do not hold at $\bar{x}$.

In nonlinear programming problems the equality of the cones $\mathcal{T}_\Omega(\bar{x})$ and $\mathcal{L}_\Omega(\bar{x})$ is sufficient to define a constraint qualification and it is known as *Abadie's Constraint Qualification.* Even a weaker version, named the equality between its polars, is also a constraint qualification and it is known as *Guignard's Constraint Qualification.* However, when we consider the (NSOCP) problem, we see that these conditions are not enough to define a constraint qualification. Also, the condition proposed by Guignard in [10] should encompass the (NSOCP), once it was proposed for problems in a general Banach spaces. Thus, let us investigate it deeper. Consider the following problem

$$\begin{array}{ll} \text{Minimize} & f(x), \\ \text{s.t.} & g(x) \in \mathbb{C}, \end{array} \tag{8}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{B}$, where $\mathbb{B}$ is a Banach space and the functions f and g are twice continuously differentiable. Thus, consider the following set

$$H(\bar{x}) \quad := \quad Dg(\bar{x})^T N_{\mathbb{C}}(g(\bar{x})) = \left\{ Dg(\bar{x})^T y \mid y \in N_{\mathbb{C}}(g(\bar{x})) \right\}, \tag{9}$$

where

$$N_{\mathbb{C}}(g(\bar{x})) \quad \doteq \quad \mathcal{T}_{\mathbb{C}}(g(\bar{x}))^{\circ} = \{ y \in \mathbb{C}^{\circ} \mid \langle g(\bar{x}), y \rangle_{\mathbb{B}} = 0 \} \tag{10}$$

is the normal cone to $\mathbb{C}$ at $g(\bar{x})$. Now we can present the main result of [10].

Theorem 3.1. *(Theorem 2 of [10]) Let $\bar{x}$ be feasible point of* (8). *Then*

i) $\mathcal{L}_{\Omega}(\bar{x}) = H(\bar{x})^{\circ}$;

ii) if $\bar{x}$ is a local minimizer and, in addition, $\mathcal{T}_{\Omega}(\bar{x})^{\circ} = \mathcal{L}_{\Omega}(\bar{x})^{\circ}$ and $H(\bar{x})$ is closed, then there exists $\overline{\mu} \in \mathbb{C}^{\circ}$ such that

$$\nabla f(\bar{x}) + Dg(\bar{x})^T \bar{\mu} = 0 \quad \text{and} \quad \langle g(\bar{x}), \bar{\mu} \rangle = 0, \tag{11}$$

that is, $\bar{x}$ is a KKT point associated to the Lagrange multiplier $\bar{\mu}$.

The theorem above explains why there is no Lagrange multiplier at Example 3.1. Even if we have the equality $\mathcal{T}_{\Omega}(\bar{x}) = \mathcal{L}_{\Omega}(\bar{x})$, this is not a CQ for a (NSOCP) problem. This misunderstanding probably happens due to a bias from nonlinear programming problems. Indeed, when we consider the non-negative orthant at NLP problems, this cone is polyhedral and then its image by a linear application is also polyhedral, which means that it is closed as Guignard request. On the other hand, when we look at the second-order cone, the image of $N_{\mathbb{K}_m}(g(\bar{x}))$ by a linear application, we might have a set that is not closed. This is not trivial to verify and the interested reader can find more details in [12].

With the correct definition of Guignard's Constraint Qualification at hand, it is possible to recover the correct definition of Abadie's Constraint Qualification as well, namely, the equality $\mathcal{T}_{\Omega}(\bar{x}) = \mathcal{L}_{\Omega}(\bar{x})$ and the set $H(\bar{x})$ be closed. This definition is also presented by Börgens et al. in [9, Definition 5.5].

4 Constant rank constraint qualification for NSOCP

In this section we will present the main results of [5]. Let us introduce the definition that will be pivotal for the proposal of the constant rank-type constraint qualification.

Definition 4.1. *[5, Definition 4.1] Consider the problem* (NSOCP) *and let $\bar{x}$ be a feasible point. We say that the* facial constant rank property *holds at $\bar{x}$, if there is a neighborhood V of $\bar{x}$ such that for all subsets $J_1, J_2 \subseteq I_0(\bar{x})$, $J_3 \subseteq I_B(\bar{x})$, with $J_1 \cap J_2 = \emptyset$, and all matrices $A_j \in \mathbb{R}^{m_j \times m_j - 1}$ of full column rank with $j \in J_1$, the rank of*

$$\bigcup_{j \in J_1} \{ Dg_j(x)^T A_j \} \bigcup_{j \in J_2} \{ Dg_j(x) \} \bigcup_{j \in J_3} \{ \nabla \tilde{\phi}_j(x) \}.$$

remains constant for all $x \in V$.

The first point that has to be noticed at the definition above, is the fact that it does not have the name *constraint qualification*, because it is not a CQ. Indeed, notice that a (NSOCP) problem whose constraints are linear satisfies the facial constant rank condition. In particular, the Example 3.1 also does and it does not have a Lagrange multiplier at the local minimizer. However, with the condition presented above we can get an important result as follows:

Theorem 4.1. *[5, Theorem 4.1] Consider the problem* (NSOCP) *and let $\bar{x}$ be a feasible point. If the facial constant rank property holds at $\bar{x}$, then $\mathcal{T}_\Omega(\bar{x}) = \mathcal{L}_\Omega(\bar{x})$.*

The facial constant rank property explains in a simple way the relation between the constant rank of the constraints with the equality between the tangent and linearized cones. Since we already know that only this equality is not enough to define a CQ, like as Guignard's CQ or Abadie's CQ, the natural condition to be added to the facial condition is to request explicitly the closeness of $H(\bar{x})$. Thus, we can propose the *Constant Rank Constraint Qualification* in the following way:

Definition 4.2. *[5, Definition 4.2] Consider the problem* (NSOCP) *and let $\bar{x}$ be a feasible point. We say that the* Constant Rank Constraint Qualification (CRCQ) *holds at $\bar{x}$, if the facial constant rank property holds at $\bar{x}$ and, in addition, if the set $H(\bar{x})$ is closed.*

With this new definition at hand, we can introduce the main result obtained in [5].

Theorem 4.2. *[5, Theorem 4.2] The CRCQ condition according to the Definition 4.2 implies Abadie's CQ. In particular, the CRCQ condition is a constraint qualification for the problem* (NSOCP).

The constant rank constraint qualification proposed to (NSOCP) problems is a natural generalization of the condition known in nonlinear programming problems and contains the same well desired properties, like as the fulfillment of the linear case, it can be proved using a constant rank theorem and implies Abadies's CQ as Janin did in [11], it is independent of Robinson's Constraint Qualification and strictly weaker than the nondegeneracy condition [5, Example 4.1 and 4.2] and, moreover, it has second-order properties as it was proved in [2], as we will show later.

Before we introduce the second-order results, let us present a new constraint qualification based on curves that naturally rises from CRCQ. For such, take a direction $d \in \mathcal{L}_\Omega(\bar{x})$ and consider the following sets:

$$\begin{array}{rcl} D_B^{int}(\bar{x}) & := & \{j \in \{1,2,\ldots,q\} \mid g_j(\bar{x}) \in \mathrm{bd}^+(\mathbb{K}_{m_j}), \nabla\tilde{\phi}_j(\bar{x})^T d > 0\} \\ D_B^0(\bar{x}) & := & \{j \in \{1,2,\ldots,q\} \mid g_j(\bar{x}) \in \mathrm{bd}^+(\mathbb{K}_{m_j}), \nabla\tilde{\phi}_j(\bar{x})^T d = 0\} \\ D_0^{int}(\bar{x}) & := & \{j \in \{1,2,\ldots,q\} \mid g_j(\bar{x}) = 0, Dg_j(\bar{x})d \in \mathrm{int}(\mathbb{K}_{m_j})\} \\ D_0^0(\bar{x}) & := & \{j \in \{1,2,\ldots,q\} \mid g_j(\bar{x}) = 0, Dg_j(\bar{x})d = 0\} \\ D_0^B(\bar{x}) & := & \{j \in \{1,2,\ldots,q\} \mid g_j(\bar{x}) = 0, Dg_j(\bar{x})d \in \mathrm{bd}^+(\mathbb{K}_{m_j})\}, \end{array} \tag{12}$$

Definition 4.3. *[13, Definition 4.4.2] Consider the problem* (NSOCP) *and let $\bar{x}$ be a feasible point. Given a direction $d \in \mathcal{L}_\Omega(\bar{x})$, consider the sets defined in* (12). *We say that the* Reformulation of the McCormick *(Ref-McCormick) for* (NSOCP) *holds at $\bar{x}$ if the set $H(\bar{x})$ is closed and if there exists a twice differentiable curve $\xi : [0, \varepsilon] \to \mathbb{R}^n$ such that $\xi(0) = \bar{x}$ and $\xi'(0) = d$, and, in addition, for all $t \in (0, \varepsilon]$ we have that*

$$g_j(\xi(t)) \in \begin{cases} \mathrm{bd}^+(\mathbb{K}_{m_j}), & j \in D_0^B(\bar{x}) \cup D_B^0(\bar{x}), \\ \mathrm{int}(\mathbb{K}_{m_j}), & j \in D_0^{int}(\bar{x}) \cup D_B^{int}(\bar{x}), \\ \{0\}, & j \in D_0^0(\bar{x}). \end{cases} \tag{13}$$

The Ref-McCormick condition is a different approach to define a constraint qualification, once it is based on the existence of a curve for each direction $d \in \mathcal{L}_\Omega(\bar{x})$. This condition is weaker than CRCQ [13, Theorem 4.4.1] and implies Abadies's CQ [13, Theorem 4.4.2].

In order to present the second-order results, let us present first the following definition that associate a constraint qualification with second-order optimality conditions.

Definition 4.4. *Consider the problem* (NSOCP) *and let* $\bar{x}$ *be a KKT point associated to a Lagrange multiplier* $(\mu_1, \ldots, \mu_q)$. *We say that the* Strong Second-Order Condition (SSOC) *holds at* $(\bar{x}, \mu_1, \ldots, \mu_q)$ *if*

$$d^T \nabla^2_{xx} L(\bar{x}, \mu_1, \ldots, \mu_q) d + d^T \mathcal{H}(\bar{x}, \mu_1, \ldots, \mu_q) d \geq 0,$$

for all $d \in C(\bar{x})$, *where* $\mathcal{H}(\bar{x}, \mu_1, \ldots, \mu_q) = \sum_{j=1}^{q} \mathcal{H}_j(\bar{x}, \mu_j)$ *with*

$$\mathcal{H}_j(\bar{x}, \mu_j) := \begin{cases} -\dfrac{[\mu_j]_0}{[g_j(\bar{x})]_0} Dg_j(\bar{x})^T \Gamma_j Dg_j(\bar{x}), & \text{if } g_j(\bar{x}) \in \mathrm{bd}^+(\mathbb{K}_{m_j}), \\ 0, & \text{otherwise.} \end{cases}$$

Now let us present the following result that shows that a local minimizer that satisfies Ref-McCormick also satisfies the strong second-order condition is satisfied for any Lagrange multiplier. This result is important once it is not satisfied under Robinson's CQ and if we assume nondegeneracy condition we have that the set of Lagrange multipliers is singleton. Also keep in mind that under Ref-McCormick we might have that the set of Lagrange multipliers is not limited.

Theorem 4.3. *Let* $\bar{x}$ *be a local minimizer of the problem (NSOCP) such that Ref-McCormick holds. Then, for any Lagrange multiplier* $(\mu_1, \ldots, \mu_q)$, *we have that* $(\bar{x}, \mu_1, \ldots, \mu_q)$ *satisfies SSOC.*

To finish the results related to the second-order optimality conditions, we present the following result that shows that the Hessian of the Lagrangian function is constant for every Lagrange multiplier associated to a local minimizer $\bar{x}$ that satisfies Ref-McCormick condition.

Theorem 4.4. *Let* $\bar{x}$ *be a local minimizer of* (NSOCP) *such that Ref-McCormick holds. The quadratic form*

$$d^T \nabla^2_{xx} L(\bar{x}, \mu_1, \ldots, \mu_q) d + d^T \mathcal{H}(\bar{x}, \mu_1, \ldots, \mu_q) d \tag{14}$$

for $d \in C(\bar{x})$, *does not depend on* $(\mu_1, \ldots, \mu_q) \in \Lambda(\bar{x})$.

5 Final considerations

The study of CQ's for second-order cone programming problems has been shown a no trivial task. On the one hand we have the nondegeneracy condition and Robinson's CQ that play an important role for the theory but they have some limitations, like the fact that set of Lagrange multiplier is singleton under nondegeneracy condition and it is also the strongest CQ known, and even if the set of Lagrange multipliers be compact if we assume Robinson's condition, we do not have any interesting second-order result under this condition. On the other hand, constant rank-type constraint qualifications were proposed initially just in 2019 by Zhang and Zhang in [14]. However, in [3] we showed that all of their proposals were incorrect. The path to get the constant rank condition for (NSOCP) started with a naive proposal made in [4], where we mixed "pure" conic constraints with inequality constraints in order to get naive CQ's. After that, analyzing deeper the structure of eigenvalues and eigenvectors of the second-order cone, we could propose new conditions without using the previous one where we avoided the difficulties of a conic problem. This new approach also encompassed the so-called sequential optimality conditions and it was presented in [6]. After that, we proposed then a condition based on a constant rank theorem that reclaims the well properties of its counterpart in nonlinear programming problems and also deals with the conic difficulties, as we discussed at Section 3. These results were presented in [5]. Finally, we introduced the Ref-McCormick condition that naturally rises from CRCQ and also keep its important characteristics, such as the second-order optimality conditions obtained. Moreover, we proved that Ref-McCormick implies that the Hessian of the Lagrangian function is constant for every Lagrange multiplier, which is new even in a nonlinear programming context.

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