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Ensemble Optimal Control of Control-Affine Systems with Uncertain Initial Conditions

Oscar A. Sierra Fonseca[1], M. Soledad Aronna[2], G. de Lima Monteiro [3]
FGV-EMAp, Rio de Janeiro, RJ

Abstract. We explore optimal control problems in systems with uncertain parameters that follow a probability distribution, aiming to optimize average performance. Known as Riemann-Stieltjes, average, or ensemble optimal control, these problems are essential when parameter uncertainty plays a significant role. We establish necessary conditions and analyze feedback control structures for control-affine systems. By extending the Pontryagin Maximum Principle through a Hilbert space approach, we characterize singular arcs, and numerical examples demonstrate the practical applicability of our findings.

Keywords. Optimal Control, Uncertain Systems, Riemann-Stieltjes Problem, Pontryagin Maximum Principle, Ensemble Control.

1 Introduction

In this work, we examine optimal control problems in which parameters are described by a probability distribution, aiming to optimize control strategies based on average performance. Such problems, known as Riemann–Stieltjes optimal control, average-optimal control, or optimal ensemble control problems, arise in various applications. The study focuses on necessary conditions and feedback characterization for control-affine problems, with parameter uncertainty incorporated to enhance practical insights. We consider an uncertain initial condition, which leads to an infinite-dimensional problem. The mathematical formulation involves a dynamic system with constrained control functions, and the objective is to minimize a cost functional integrated over a probability measure. The control function belongs to an admissible set of bounded measurable functions. The study extends prior research on necessary optimality conditions, particularly for parameterized control problems. Previous studies have addressed cases with fixed initial conditions, uncertain dynamics, and penalized optimization approaches. Notable contributions include applications in aerospace, reinforcement learning, and numerical methods. The Pontryagin Maximum Principle (PMP) has been used to establish necessary conditions for optimality in various settings.

This work builds on previous results by developing PMP conditions for problems with uncertain initial conditions. The analysis treats the initial condition as an element of a Hilbert space, accommodating infinite-dimensional settings. The results characterize singular arcs in feedback form for scalar and vector-valued controls with commuting vector fields. Finally, numerical examples illustrate the effectiveness and accuracy of the proposed approach.

[1]oscar.fonseca@fgv.br
[2]soledad.aronna@fgv.br
[3]g.delimamonteiro@gmail.com

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2 The Problem

We consider a Riemann-Stieltjes control problem when the initial data includes uncertain parameters, it results in a formulation within an infinite-dimensional space. More precisely, we are dealing with the following problem

$$(P)\qquad \begin{aligned} &\text{minimize} \quad J[u(\cdot)] := \int_\Omega g(x(T,\omega),\omega)\, d\mu(\omega),\\ &\text{s.t.}\\ &\begin{cases} \dot{x}(t,\omega) = f_0(x,\omega) + \displaystyle\sum_{i=1}^{m} f_i(x,\omega)u_i(t), \\ \qquad\qquad\qquad\qquad \text{a.e. } t \in [t_0,T],\ \omega \in \Omega, \\ x(t_0,\omega) = \varphi(\omega), \qquad \omega \in \Omega, \\ u(t) \in U(t), \qquad \text{a.e. } t \in [t_0,T], \end{cases} \end{aligned} \tag{1}$$

where $f_i \colon \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ for $i = 0,\ldots,m$, $g \colon \mathbb{R}^n \times \Omega \to \mathbb{R}$ and $u : [t_0,T] \to \mathbb{R}^m$ is the control function, which belongs to the *admissible controls set*

$$\mathcal{U}_{\mathrm{ad}}[t_0,T] := \{u \in L^\infty(t_0,T;\mathbb{R}^m) : u(t) \in U(t) \quad \text{a.e. } t \in [t_0,T]\}, \tag{2}$$

where $U : [0,T] \rightsquigarrow \mathbb{R}^m$ is a multifunction taking non-empty and closed values contained in a compact subset $\mathbf{U} \subset \mathbb{R}^m$. We investigate necessary optimality condition for problem (P) with initial condition φ, where $\varphi : \Omega \to \mathbb{R}^n$ is a measurable function, *i.e.,* we suppose that φ belongs to the space of measurable functions $L^2(\mu,\Omega;\mathbb{R}^n)$, given by

$$L^2(\mu,\Omega;\mathbb{R}^n) := \Big\{\varphi : \Omega \to \mathbb{R}^n : \int_\Omega |\varphi(\omega)|^2\, d\mu(\omega) < \infty\Big\},$$

which is a Hilbert space endowed with the scalar product

$$\langle \varphi, \psi\rangle := \int_\Omega \varphi(\omega)\cdot\psi(\omega)\, d\mu(\omega) \quad \text{for} \quad \varphi,\psi \in L^2(\mu,\Omega;\mathbb{R}^n).$$

We will use the notation $\|\cdot\|_{L^2}$ to refer to the norm associated with the above scalar product. Here, we redefine the dynamics as a map taking values in an infinite-dimensional space, more precisely, we define the dynamics $f : L^2(\mu,\Omega;\mathbb{R}^n) \times \mathbb{R}^m \to L^2(\mu,\Omega;\mathbb{R}^n)$ as

$$f(\varphi(\cdot),u) := f_0(\varphi(\cdot),\cdot) + \sum_{i=1}^{m} f_i(\varphi(\cdot),\cdot)u_i. \tag{3}$$

2.1 Preliminaries

We will adopt the following set of assumptions, which we refer to hereafter as (H0):

(i) μ is a probability measure defined over a complete separable metric space (Ω,ρ_Ω) and, for a.e $t \in [t_0,T]$, $U(t)$ is a non-empty, compact and convex subset of $\mathbb{R}^m$ and its graph $GrU(\cdot)$ is a $\mathcal{L} \times \mathcal{B}^m$-measurable set.

(ii) For every $i = 0, 1, ..., m$, there exist constants $c_i, k_i > 0$ such that, for every $x, x' \in \mathbb{R}^n$, $\omega \in \Omega$, f_i satisfies

$$|f_i(x, \omega)| \leq c_i(1 + |x|) \tag{4}$$

and

$$|f_i(x, \omega) - f_i(x', \omega)| \leq k_i|x - x'|, \quad \text{for all} \quad i = 0, 1, ..., m. \tag{5}$$

(iii) The function $g : \mathbb{R}^n \times \Omega \to \mathbb{R}$ is $\mathcal{B}^n \times \mathcal{B}_\Omega$-measurable and there exist positive constants $k_g \geq 1$ and M such that, for all $x, x' \in \mathbb{R}^n$, $\omega \in \Omega$,

$$\begin{cases} |g(x, \omega)| \leq M, \\ |g(x, \omega) - g(x', \omega)| \leq k_g|x - x'|. \end{cases} \tag{6}$$

An *admissible process* $(u, \{x(., \omega) : \omega \in \Omega\})$ consists of a control function u belonging to $\mathcal{U}_{ad}[t_0, T]$, coupled with its associated family of arcs $\{x(., \omega) \in W^{1,1}([0, T], \mathbb{R}^n) : \omega \in \Omega\}$ such that, for each $\omega \in \Omega$, $x(\cdot, \omega)$ is solution of the following evolution integral equation

$$x(t, \omega) = \varphi(\omega) + \int_{t_0}^{t} f_0(x(\sigma, \omega), \omega) + \sum_{i=1}^{m} f_i(x(\sigma, \omega), \omega)\, u_i(\sigma)\, d\sigma. \tag{7}$$

If there exists a process $(\bar{u}, \{\bar{x}(., \omega) : \omega \in \Omega\})$ solving problem (P), then it is called *optimal pair*, and we will refer to $\bar{x}$ and $\bar{u}$ as *optimal trajectory* and *control*, respectively.

Definition 2.1 ($W^{1,1}$-local minimizer). *A process* $(\bar{u}, \{\bar{x}(\cdot, \omega) : \omega \in \Omega\})$*, is said to be an* $W^{1,1}$-local minimizer *for* (P) *if there exists* $\epsilon > 0$ *such that*

$$\int_\Omega g(\bar{x}(T, \omega), \omega) d\mu(\omega) \leq \int_\Omega g(x(T, \omega), \omega) d\mu(\omega),$$

for all admissible processes $(u, \{x(\cdot, \omega) : \omega \in \Omega\})$ *such that*

$$\|\bar{x}(., \omega) - x(., \omega)\|_{W^{1,1}([0,T];\mathbb{R}^n)} \leq \epsilon\ ,\ \forall \omega \in \mathit{supp}(\mu).$$

We recall the assumptions and results established by Bettiol-Khalil in [3]. Given a $W^{1,1}$-local minimizer $(\bar{u}, \{\bar{x}(\cdot, \omega) : \omega \in \Omega\})$ and $\delta > 0$, we shall refer to the following set of assumptions as (H1):

(i) There is a modulus of continuity $\theta_f : [0, \infty) \to [0, \infty)$ such that, for all $\omega, \omega_1, \omega_2 \in \Omega$,

$$\int_{t_0}^{T} \sup_{x \in \bar{x}(t,\omega) + \delta\mathbb{B}, u \in U(t)} |f(t, x, u, \omega_1) - f(t, x, u, \omega_2)| dt \leq \theta_f(\rho_\Omega(\omega_1, \omega_2))$$

(ii) $g(., \omega)$ is differentiable on $\bar{x}(T, \omega) + \delta\mathbb{B}$, for each $\omega \in \Omega$, $\nabla_x g(., \omega)$ is continuous on $\bar{x}(T, \omega) + \delta\mathbb{B}$ and $\nabla_x g(x, \cdot)$ is continuous on Ω;

(iii) $x \mapsto f(t, x, u, \omega)$ is continuously differentiable on $\bar{x}(t, \omega) + \delta\mathbb{B}$ for all $u \in U(t)$, $\omega \in \Omega$ a.e. $t \in [t_0, T]$, and $\omega \mapsto \nabla_x f(t, x, u, \omega)$ is continuous uniformly with respect to (t, x, u).

(iv) We assume that the functions f_i are twice continuously differentiable on $\bar{x}(t, \omega) + \delta\mathbb{B}$ for all $\omega \in \Omega$ and a.e. $t \in [t_0, T]$ and there exists $C_f^k > 0$ such that, for $i \in \{0, \ldots, m\}, k = 1, 2$,

$$\left| \frac{\partial^k f_i}{\partial x^k}(\bar{x}(t, \omega), \omega) \right| < C_{f_i}^k, \quad \omega \in \text{supp}(\mu),\ \text{a.e.}\ t \in [0, T];$$

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3 Necessary Optimality Conditions

3.1 Necessary Conditions

Before studying the necessary optimality conditions for problem (P) with an uncertain initial condition $\varphi \in L^2(\mu, \Omega; \mathbb{R}^n)$, we first establish the existence of an optimal control (see [2]). Then, using technical results, we derive the Pontryagin Maximum Principle (PMP) in Theorem 3.1. We begin with the following result.

Lemma 3.1. *Suppose that g is differentiable with respect to $\bar{x}$, and that $\nabla_x g(\cdot,\cdot)$ is continuous. Set $\delta > 0$. Then for all $\varphi \in \bar{x}(T,\cdot) + \delta\mathbb{B}$, the functional*

$$\mathcal{J}(\varphi) := \int_\Omega g(\varphi(\omega), \omega) d\mu(\omega)$$

is Fréchet differentiable in φ, and the derivative of $\mathcal{J}$ in φ is the functional $D\mathcal{J}(\varphi) \in L^2(\mu, \Omega; \mathbb{R}^n)^*$ given by

$$D\mathcal{J}(\varphi)\psi = \int_\Omega \nabla_x g(\varphi(\omega), \omega) \cdot \psi(\omega)\, d\mu(\omega). \tag{8}$$

Note that, by the Riesz Representation Theorem, $\nabla_x g(\varphi(\cdot),\cdot) \in L^2(\mu, \Omega; \mathbb{R}^n)$ is the unique representative of the derivative of $\mathcal{J}$. Thus, we denote the derivative as $D\mathcal{J}(\varphi) := \nabla_x g(\varphi)$.

The following result on necessary conditions is based on those obtained in [4].

Theorem 3.1. *Assume that assumptions (H0) and (H1) hold, and that f is Fréchet differentiable with respect to x. Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$-*local minimizer *for (P) and set the functional $p := \nabla_x g(\bar{x}(T))$. Then, the solution of the backward problem*

$$\begin{cases} -\bar{p}'(t) = \left(\dfrac{\partial f}{\partial x}(\bar{x}(t), \bar{u}(t))\right)^* \bar{p}(t), \quad t \in [t_0, T] \\ -\bar{p}(T) = p. \end{cases} \tag{9}$$

satisfies the Maximum Principle

$$\int_\Omega p(t,\omega) f(\bar{x}(t,\omega), \bar{u}(t), \omega) d\mu(\omega) = \max_{u \in U(t)} \int_\Omega p(t,\omega) f(\bar{x}(t,\omega), u, \omega) d\mu(\omega) \quad a.e\ t \in [0,T]. \tag{10}$$

4 Scalar Control Variable

In this section, we assume that the control variable is scalar, *i.e.,* $m = 1$. Given two smooth vector fields $f, g : \mathbb{R}^n \to \mathbb{R}^n$, the *Lie bracket* $[f, g]$ is a new vector field defined as:

$$[f, g] := g'f - f'g.$$

Theorem 4.1 (Characterization of the optimal control for the scalar control-affine case)**.** *Let Assumptions (H0) and (H1) hold. Then, there exists a $W^{1,1}$-minimizer $(\bar{u}, \{\bar{x}(.,\omega) : \omega \in \Omega\})$ with p as in Theorem 3.1. Additionally, the optimal control $\bar{u}$ satisfies the conditions*

$$\bar{u}(t) = \begin{cases} u_{\max} & \text{if } \Psi(t) > 0, \\ u_{\min} & \text{if } \Psi(t) < 0, \\ \text{singular} & \text{if } \Psi(t) = 0, \end{cases} \tag{11}$$

where Ψ *is the* switching function *that is given by*

$$\Psi(t) = \int_\Omega p(t,\omega) f_1(\bar{x}(t),\omega) d\mu(\omega). \tag{12}$$

Moreover, over singular arcs, the control $\bar{u}$ *satisfies*

$$\int_\Omega p[f_0,[f_0,f_1]]d\mu(\omega) + \bar{u}\int_\Omega p[f_1,[f_0,f_1]]d\mu(\omega) = 0. \tag{13}$$

5 Numerical Examples

5.1 Numerical Scheme

In order to numerically solve the problem (P), we use the *sample average approximation method.* Specifically, at each iteration k a finite random set $\Omega_k = \{\omega_i^k\}_{i=1}^k$ of independent and identically μ-distributed samplings is selected from the parameter space Ω and the following a classical finite-dimensional optimal control problem is considered:

$$(P)_k \quad \begin{aligned} &\text{minimize} \quad J_k[u(\cdot)] = \frac{1}{k}\sum_{i=1}^{k} g(x_i(T,\omega_i^k),\omega_i^k),\\ &\text{s.t.}\\ &\begin{cases} \dot{x}(t,\omega_i^k) = f_0(x_i(t,\omega_i^k),\omega_i^k) + \sum_{i=1}^{m} f_i(x_i(t,\omega_i^k),\omega_i^k)u_i(t),\\ x(0,\omega_i^k) = \varphi(\omega_i^k), \quad i=1,\dots,k. \end{cases} \end{aligned} \tag{14}$$

In the case of known initial condition x_0, one set $\varphi(\omega_i^k) = x_0$, for $i = 1,\dots,k$. In this section, we assume that the set of admissible controls is defined by

$$\hat{\mathcal{U}}_{\text{ad}} := \left\{u \in L^\infty(t_0,T;\mathbb{R}^m) \cap L^2(0,T;\mathbb{R}^m) : u(t) \in U(t) \quad \text{a.e. } t\in[0,T]\right\}.$$

Note that, $\hat{\mathcal{U}}_{\text{ad}}$ is a complete, separable metric space with respect to the L^2-topology. Consequently, we can apply the numerical scheme developed in [6].

5.2 Example: Optimizing Fishing Strategies

In this subsection, we consider a modified version of the optimal fishing problem presented in [1] as an illustrative example. Let $x(t,\omega)$ denote the size of the halibut fish population at time t, corresponding to ω. The control $u(t) \in \mathbb{R}$ represents the fishing effort per unit time, and the objective is to maximize the average fishing revenue over the fixed time interval $[0,T]$. We obtain the following model:

$$\begin{aligned} &\text{maximize } J[u(\cdot)] = \int_\Omega\int_0^T \left(E - \frac{c}{x(t,\omega)}\right) u(t)U_{\max}\, dt\, d\mu(\omega)\\ &\text{s.t.}\\ &\begin{cases} \dot{x}(t,\omega) = r(\omega)x(t,\omega)\left(1 - \frac{x(t,\omega)}{k(\omega)}\right) - u(t)U_{\max} \quad \text{a.e. on } [0,T],\\ 0 \le u(t) \le 1 \quad \text{a.e. on } [0,T],\\ x(0,\omega) = \varphi(\omega) \end{cases} \end{aligned} \tag{15}$$

with fixed parameters $T = 10$, $E = 1$, $c = 17.5$, $U_{\max} = 20$; and stochastic parameters $\varphi \sim \mathcal{TN}(70, 5, 40, 90)$, $r \sim \mathcal{TN}(0.71, 0.05, 0.1, 1)$, $k \sim \mathcal{TN}(80.5, 10, 65, 95)$, where $\mathcal{TN}(\mu, \sigma, a, b)$ denotes the truncated normal distribution with mean μ, standard deviation σ, and support in the interval $[a, b]$, as in [5].

From Theorem 4.1, we obtain an explicit formula for the optimal control of problem (15), which is then compared with the solution obtained numerically using Python (Figura 1).

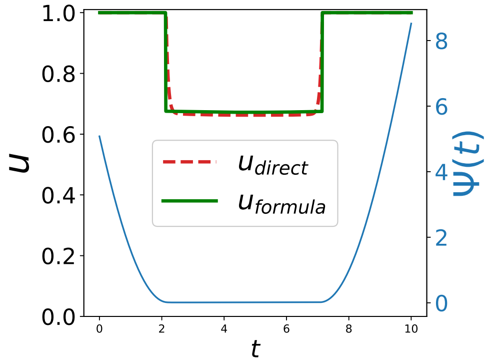

Figura 1: In green, optimal control calculated with the formula derived in theorem 4.1. In dashed red, the optimal control is calculated with the GEKKO library. In blue, the switching function Ψ given by (12). Source: Authors.

The graphs of the relative cost distance (Figura 2) and the relative control distance (Figura 3) between successive iterations exhibit a decreasing trend toward zero, indicating convergence and stability of the proposed method.

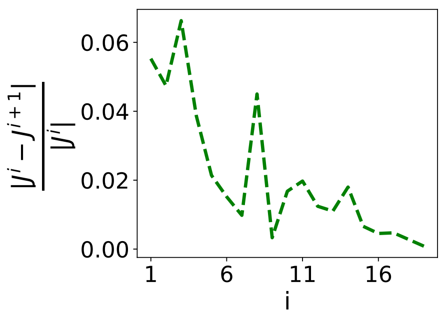

Figura 2: Cost relative distance between successive iterations. Source: Authors.

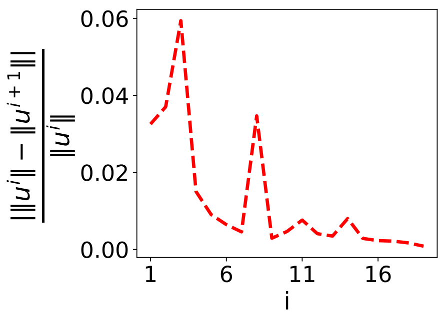

Figura 3: Control relative distance between successive iterations. Source: Authors.

6 Conclusions

In this work, we explicitly characterize the optimal control for the Riemann–Stieltjes optimal control problem. To this end, we apply Pontryagin's Maximum Principle to derive necessary

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conditions for optimality. Our approach is formulated in an infinite-dimensional space, as the initial condition can be treated as an element of a Hilbert space. This enables a rigorous analysis of the control structure under uncertainty.

Numerical results confirm the effectiveness of our approach. The resulting control trajectories closely match those obtained using Python's GEKKO optimization package, validating our theoretical findings (Figura 1). Notably, the iterative reduction in both relative cost (Figura 2) and control distances (Figura 3), converging toward zero, highlights the stability and convergence of the proposed method. These findings demonstrate the robustness of the framework and its potential applicability to a broad class of uncertain optimal control problems. Future work may explore extensions to more general system dynamics and alternative numerical solution techniques

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