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Lyapunov Functions for a Recation-Diffusion Model of Leslie-Gower Type

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Abstract. The aim of this work is to construct Lyapunov functions to extend the global stability results of a Leslie-Gower reaction-diffusion model in a bounded domain $\Omega \in \mathbb{R}^n$, with no-flux boundary condition. We extended the global stability results to other parameter regions of the reaction-diffusion model using these Lyapunov functions.

Keyword. Global Stability, Lyapunov Function, Leslie Gower Model, Reaction-Diffusion, Predator-Prey

1 Introduction

In 1948 and 1960, Leslie and Gower proposed a model that considers the response of the predator to the prey density by considering the environmental carrying capacity proportional to prey abundance [3], [4]. By scaling, the model is the following one:

$$\begin{aligned} \frac{du}{dt} &= u(\lambda - \alpha u - \beta v), \\ \frac{dv}{dt} &= \mu v \left(1 - \frac{v}{u}\right). \end{aligned} \tag{1}$$

The spatial dynamic associated with this model was studied in several works [1], [2], [5], [7]. The reaction-diffusion model with diffusion of both species is the following:

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + u(\lambda - \alpha u - \beta v), && x \in \Omega, t > 0 \\ \frac{\partial v}{\partial t} &= d_2 \Delta v + \mu v \left(1 - \frac{v}{u}\right), && x \in \Omega, t > 0 \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, && x \in \partial\Omega, t > 0 \\ u(x,0) &= u_0(x) > 0, v(x,0) = v_0(x) > 0, && x \in \Omega \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$ and d_1, d_2 are the positive diffusion coefficients of the prey and predator populations, respectively.

The system (2) has a unique constant positive equilibrium (u^*, v^*):

$$(u^*, v^*) = \left(\frac{\lambda}{\alpha + \beta}, \frac{\lambda}{\alpha + \beta}\right). \tag{3}$$

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The equilibrium (3) is globally asymptotically stable for the ordinary differential system (1), when all parameters are positive. However, the global asymptotical stability of this equilibrium for the associated reaction diffusion system (2), it is only a conjecture. By constructing a Lyapunov function, Du and Hsu [1] and Hsu [2] proved that for $\alpha > \beta$, the equilibrium is globally asymptotically stable.

Theorem 1.1. ***(Du-Hsu)*** *Suppose $\alpha > \beta$, then (u^*, v^*) is globally asymptotically stable for system (2) in the sense that every positive solution satisfies*

$$\lim_{t \longrightarrow \infty} (u(x,t), v(x,t)) = (u^*, v^*)$$

uniformly on $\overline{\Omega}$.

Nevertheless, when $\alpha \leq \beta$, it is an open question. In this case, some results were obtained in several works, such as Du and Hsu [1], Hsu [2]. The following Theorem was established in the work of Du an Hsu [1]:

Theorem 1.2. ***(Du-Hsu)*** *Suppose $\alpha/\beta > s_0 \in \left(\frac{1}{4}, \frac{1}{5}\right)$ where s_0 is the unique positive root of the three-degree polynomial $h(s) = 32s^3 + +16s^2 - s - 1,\ s \in \mathbb{R}$, then (u^*, v^*) is globally asymptotically stable for system (2).*

We modify the Lyapunov function proposed by Du and Hsu to obtain a new one and extend these results to the following Theorem:

Theorem 1.3. *Suppose $d_1 = d_2 = d$ for some constant $d > 0$, and $\lambda, \mu, \alpha, \beta$ are positive constant. If $\dfrac{\mu}{\lambda} > \dfrac{\beta}{\alpha}\left(1 + \dfrac{\beta}{\alpha}\right)^2 \left(\dfrac{\beta}{\alpha} - 1\right)$, then (u^*, v^*) is globally asymptotically stable for the system (2).*

A generalization of the Leslie-Gower model was proposed by Qi-Zhou [5], considering a more general reaction term. The model is the following:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= d_1 \Delta u + u(\lambda - \alpha u^\sigma - \beta v), && x \in \Omega, t > 0\\
\frac{\partial v}{\partial t} &= d_2 \Delta v + \mu v\left(1 - \frac{v}{u^\sigma}\right), && x \in \Omega, t > 0\\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, && x \in \partial\Omega, t > 0\\
u(x,0) &= u_0(x) > 0, v(x,0) = v_0(x) > 0, && x \in \Omega
\end{aligned} \tag{4}$$

where the new parameter is the positive constant σ, which measures intraspecific competition among prey and the effect on predator carrying capacity. In preys, a modified logistic growth $u(\lambda - \alpha u^\sigma)$ is considered. In predators, the logistic growth with modified carrying capacity is $\mu v(1 - \dfrac{v}{u^\sigma})$. The system (4) has a unique constant positive equilibrium (u^*, v^*):

$$(u^*, v^*) = \left(\left(\frac{\lambda}{\alpha + \beta}\right)^{(1/\sigma)}, \frac{\lambda}{\alpha + \beta}\right). \tag{5}$$

Qi and Zhu [5], combining the transformation technique and comparison principle, proved the following result:

Theorem 1.4. ***(Qi-Zhu)*** *Suppose $d_1 = d_2 > 0, 0 < \sigma < 1$, and $\lambda, \mu, \alpha, \beta$ are positive constants. Then, (u^*, v^*) is globally asymptotically stable if $\dfrac{\mu}{\lambda} > \sigma\dfrac{\beta}{\alpha}$.*

Zhou and Wei [7], by the transformation technique and Lyapunov function method, extend the result to the following:

Theorem 1.5. ***(Zhou-Wei)*** *Suppose $\lambda, \mu, \alpha, \beta$ are positive constants. Then, (u^*, v^*) is globally asymptotically stable if $\sigma > \dfrac{\beta}{\alpha}$.*

and

Theorem 1.6. ***(Zhou-Wei)*** *Suppose $d_1 = d_2 = d > 0, \sigma > 1$, and $\lambda, \mu, \alpha, \beta$ are positive constants. Then, (u^*, v^*) is globally asymptotically stable if $\dfrac{\alpha}{\beta} > (1+\sigma)\dfrac{\lambda}{\mu} - 1$.*

We extend the parameter region of global stability for the model by constructing a new Lyapunov function. Our method is based on the construction of the Lyapunov function by modifying a previous one. We establish the following Theorem:

Theorem 1.7. *Suppose $d_1 = d_2 = d$ for some constant $d > 0$, $\sigma > 1$, and $\lambda, \mu, \alpha, \beta$ are positive constant. If $\dfrac{\mu}{\lambda} > \dfrac{\beta}{\alpha}\left(1 + \dfrac{\beta}{\alpha}\right)^2 \left(\dfrac{\beta}{\alpha} - \sigma\right)$, then (u^*, v^*) is globally asymptotically stable for the system (4).*

The current work is organized as follows. In Section 2, we extend the global stability results obtained by Du and Hsu [1], Hsu [2], for the Leslie-Gower model. We extend the global stability results for the generalized Leslie-Gower model in Section 3. Finally, our conclusions are given in Section 4.

2 Global Stability of the Leslie-Gower Model

Let $u = (u_1, u_2, \ldots, u_n)^T$ a non-negative vector of concentrations and the system of ordinary differential equations

$$\frac{du}{dt} = f(u), \tag{6}$$

where $f : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n$ is a C^1 function, $f(u) = (f_1(u), \ldots, f_n(u))^T$.

Let $\Omega \subset \mathbb{R}^N$ a bounded domain with smooth boundary $\partial\Omega$, and ν is the outward unit normal vector on $\partial\Omega$. The ODE system (6) is the corresponding reaction term of a reaction-diffusion system given by

$$\begin{aligned} &\frac{\partial u}{\partial t} = D\Delta u + f(u), && x \in \Omega, t > 0 \\ &\frac{\partial u}{\partial \nu} = 0, && x \in \partial\Omega, t > 0 \\ &u(x,0) = u_0(x) && x \in \Omega \end{aligned} \tag{7}$$

where D is a diagonal matrix with nonnegative entries, $D = diag(d_1, \ldots, d_n)$, $d_i > 0, i = 1, \ldots, n$, $x \in \Omega \subset \mathbb{R}^N$, $t \in [0, \infty)$. Let $V : G \subset \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ a Lyapunov function on a open subset G and $u^* \in \mathbb{R}^n_+$ the equilibrium of the system (6). This Lyapunov function $V(u)$ for the ODE system (6) allows us to introduce a Lyapunov functional as in Hsu [2], Redheffer *et al.* [6] for the reaction-diffusion system (7).

$$E(t) = \int_\Omega V(u(x,t))dx. \tag{8}$$

The following Theorem establishes a global stability criterion for the reaction-diffusion system.

Theorem 2.1. *Let $u^* \in G \subset \mathbb{R}^n_+$ be an equilibrium of the ordinary differential system (6) globally asymptotically stable and $V(u)$ a Lyapunov function.*

If DH is a positive definite matrix, where $D = diag(d_1, d_2, \ldots, d_n), d_i > 0, i = 1, \ldots, n$ is the diagonal matrix of the diffusion coefficients, and $H = \left(\frac{\partial^2 V}{\partial u_i \partial u_j}\right)$ is the Hessian matrix of V then $E(t)$ defined by (8) is a Lyapunov functional for the associated reaction-diffusion system (7) and all bounded solutions $u(x,t)$ that remain in G (for $t \geq 0$) approach u^* as $t \longrightarrow \infty$.

The proof of **Theorem 2.1** could be found in Hsu [2], for $D_i = D_j$ for all $i, j = 1, \ldots, n$. A proof without this condition could be found in Redheffer *et al.*, however, with the same selfadjoint elliptic operator of second order for all components of u, assuming that the operator is uniformly elliptic:

$$\sum_{i,j}^{n} a_{ij}(x,u) \left(\frac{\partial^2 V}{\partial u_i \partial u_j}\right) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \geq \theta \left|\nabla u\right|^2, \theta > 0.$$

When $a_{ij}(x,u)\left(\frac{\partial^2 V}{\partial u_i \partial u_j}\right)$ is symetric, it is equivalent to being positive definite.

We modify the Lyapunov function $V(u,v)$ used by Du and Hsu [1] for the system of ordinary differential equations (1) to obtain a new one and improve the stability results of the reaction-diffusion system.

$$V(u,v) = \int_{u^*}^{u} \frac{(\xi - u^*)}{\xi^2} d\xi + c \int_{v^*}^{v} \frac{(\eta - v^*)}{\eta} d\eta, \text{ where } c = (\beta/\mu).$$

We define the following nonnegative function:

$$W(u,v) = Exp\left[V(u,v)\right] - 1 = Exp\left[\int_{u^*}^{u} \frac{(\xi - u^*)}{\xi^2} d\xi + c \int_{v^*}^{v} \frac{(\eta - v^*)}{\eta} d\eta\right] - 1, \quad (9)$$

where $c = (\beta/\mu)$. We have the following proposition:

Proposition 2.1. *$W(u,v)$ is a Lyapunov function for the ordinary system.*

Proof The Lyapunov function $V(u,v)$ proposed by Du and Hsu [1] satisfies

$$\dot{V}(u,v) = \frac{\partial V}{\partial u}\frac{du}{dt} + \frac{\partial V}{\partial v}\frac{dv}{dt} =$$

$$= \frac{u - u^*}{u}(\lambda - \alpha u - \beta v) + c\mu (v - v^*)\left(1 - \frac{v}{u}\right) =$$

$$= \frac{u - u^*}{u}(\alpha u^* + \beta v^* - \alpha u - \beta v) + c\mu (v - v^*)\frac{u - u^* + v^* - v}{u} =$$

$$= -\alpha \frac{(u - u^*)^2}{u} + (c\mu - \beta)\frac{(u - u^*)(v - v^*)}{u} - c\mu \frac{(v - v^*)^2}{u} =$$

$$= -\alpha \frac{(u - u^*)^2}{u} - \beta \frac{(v - v^*)^2}{u} \leq 0.$$

Hence, the function $W(u,v)$ satisfies

$$\dot{W}(u,v) = \frac{\partial W}{\partial u}\frac{du}{dt} + \frac{\partial W}{\partial v}\frac{dv}{dt} = Exp[V(u,v)]\frac{\partial V}{\partial u}\frac{du}{dt} + Exp[V(u,v)]\;\frac{\partial V}{\partial v}\frac{dv}{dt}$$

$$= -Exp[V(u,v)]\left[\alpha\frac{(u-u^*)^2}{u} + \beta\frac{(v-v^*)^2}{u}\right] \leq 0,$$

and equality holds only if $(u,v) = (u^*, v^*)$. □

First, we remark on a necessary fact for global stability, the boundedness of the solutions obtained in [1] and [2].

$$\overline{\lim_{t\longrightarrow\infty}} u(x,t) \leq \lambda/\alpha,\ \overline{\lim_{t\longrightarrow\infty}} v(x,t) \leq \lambda/\alpha.$$

To improve the stability results, we use the modified Lyapunov function for the ordinary differential system defined by (9).

We need the following lemma which can be proved using the boundedness of the solution:

Lemma 2.1. *Suppose* $\dfrac{\mu}{\lambda} > \dfrac{\beta}{\alpha}\left(1+\dfrac{\beta}{\alpha}\right)^2\left(\dfrac{\beta}{\alpha}-1\right)$, *then exist* $T_0 > 0$ *such that* $(u^*)^2v^* > c(v-v^*)^2\left(u^2 - 2u^*u\right)$, *for all* $u > 0, v > 0$, *and for all* $t \geq T_0 > 0$.

Proof of Theorem 1.3 We consider the Lyapunov functional

$$E(t) = \int_\Omega W(u(x,t))dx,$$

associated to the Lyapunov function of the ordinary differential system $W(u,v) = Exp[V(u,v)]-1$, where $c = (\beta/\mu)$.

By **Theorem 2.1**, we need to prove that $D\left(\frac{\partial^2 W}{\partial u \partial v}\right)$ is a positive definite matrix.

Then, it is sufficient to prove that $W_{uu} > 0$ and $W_{uu}W_{vv} - W_{uv}^2 > 0$. After some calculations, we obtain the following results:

$$W_{uu} = \left[V_u^2 + V_{uu}\right]\ Exp[V(u,v)] = \frac{(u^*)^2}{u^4}\ Exp[V(u,v)] > 0,$$

$$W_{uv} = W_{vu} = V_uV_v\ Exp[V(u,v)] = \frac{(u-u^*)}{u^2}\frac{c(v-v^*)}{v}\ Exp[V(u,v)],$$

$$W_{vv} = \left[V_v^2 + V_{vv}\right]\ Exp[V(u,v)] = \left[c^2\frac{(v-v^*)^2}{v^2} + \frac{cv^*}{v^2}\right]\ Exp[V(u,v)] > 0.$$

Hence, the determinant is $W_{uu}W_{vv} - W_{uv}^2 =$

$$= Exp[V(u,v)]^2\left[\frac{c^2(u^*)^2}{u^4}\frac{(v-v^*)^2}{v^2} + \frac{cv^*}{v^2}\frac{(u^*)^2}{u^4} - \frac{(u-u^*)^2}{u^4}\frac{c^2(v-v^*)^2}{v^2}\right] =$$

$$= Exp[V(u,v)]^2\left[\frac{c^2(v-v^*)^2\left(-u^2+2u^*u\right) + c(u^*)^2v^*}{u^4v^2}\right] > 0,$$

by Lemma (2.1). Therefore, $D\left(\frac{\partial^2 W}{\partial u \partial v}\right)$ is positive definite, and $E(t)$ is a Lyapunov functional for the reaction-diffusion system. The equilibrium (u^*, v^*) is globally asymptotically stable. □

3 The Leslie-Gower Model with General Reaction Term

In this section, we extend the global stability results of the Leslie-Gower model with a general reaction term, given by system (4), by constructing a new Lyapunov function. By the comparison theorems, we obtained the boundedness of the solutions:

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$$\overline{\lim_{t\longrightarrow\infty}} u(x,t) \leq (\lambda/\alpha)^{1/\sigma}\,,\ \overline{\lim_{t\longrightarrow\infty}} v(x,t) \leq \lambda/\alpha.$$

We propose a new Lyapunov function, adapting the one proposed by Zhou and Wei [7], in the same way as the Leslie-Gower model:

Proposition 3.1. *The function*

$$W^\sigma(u,v) = Exp[V^\sigma(u,v)] - 1 = Exp\left[\int_{u^*}^{u} \frac{(\xi^\sigma - (u^*)^\sigma)}{\xi^{1+\sigma}} d\xi + c\int_{v^*}^{v} \frac{(\eta - v^*)}{\eta} d\eta\right] - 1,$$

where $c = (\beta/\mu)$ is a Lyapunov function for the ordinary system associated to (4).

Proof The function $W^\sigma(u,v)$ satisfies:

$$\dot{W^\sigma}(u,v) = -Exp[V^\sigma(u,v)]\left[\alpha\frac{(u^\sigma - (u^*)^\sigma)^2}{u^\sigma} + \beta\frac{(v-v^*)^2}{u^\sigma}\right] \leq 0,$$

and equality holds only if $(u,v) = (u^*,v^*)$. □

To extend the global stability results of the Leslie-Gower model with general reaction term, we need the following lemma, which can be proved using the boundedness of the solutions:

Lemma 3.1. *Suppose* $\dfrac{\mu}{\lambda} > \dfrac{\beta}{\alpha}\left(1+\dfrac{\beta}{\alpha}\right)^2\left(\dfrac{\beta}{\alpha} - \sigma\right)$, *then there exists $T_0 > 0$ such that* $(u^*)^{2\sigma} v^* > c(v-v^*)^2\left(u^\sigma - (1+\sigma)(u^*)^\sigma\right)u^\sigma$, *for all $u > 0, v > 0$, and for all $t \geq T_0 > 0$.*

Proof of Theorem 1.7

The proof of **Theorem 1.7** is similar to **Theorem 1.3** and we omit some details. We consider the associated Lyapunov functional of the ordinary differential system:

$$E^\sigma(t) = \int_\Omega W^\sigma(u(x,t))dx.$$

By **Theorem 2.1**, we need to prove that $D\left(\frac{\partial^2 W^\sigma}{\partial u \partial v}\right)$ is a positive definite matrix. It is sufficient to prove that $W^\sigma_{uu} > 0$ and $W^\sigma_{uu}W^\sigma_{vv} - (W^\sigma_{uv})^2 > 0$. After some calculations, we obtain the following results.

$W^\sigma_{uu} = \ Exp[V^\sigma(u,v)]\dfrac{(\sigma-1)u^\sigma(u^*)^\sigma + (u^*)^{2\sigma}}{u^{2\sigma+2}} > 0$, since $\sigma > 1$,

and the determinant is $W^\sigma_{uu}W^\sigma_{vv} - (W^\sigma_{uv})^2 =$

$$= Exp[V^\sigma(u,v)]^2\left[\frac{c^2(v-v^*)^2\left[-u^{2\sigma} + (1+\sigma)(u^*)^\sigma u^\sigma\right] + cv^*\left[(u^*)^{2\sigma} + (\sigma-1)u^\sigma u^*)^\sigma\right]}{u^{2\sigma+2}v^2}\right] > 0,$$

by Lemma (3.1) and $\sigma > 1$. Therefore, $D\left(\frac{\partial^2 W^\sigma}{\partial u \partial v}\right)$ is positive definite, and $E^\sigma(t)$ is a Lyapunov functional for the reaction-diffusion system. The equilibrium (u^*,v^*) is globally asymptotically stable in the positive region. □

4 Discussion and Conclusion

In this work, we propose a method based on constructing new Lyapunov functions for a reaction diffusion system by modifying a previous one. Given a Lyapunov function $V(u,v)$, we define the Lyapunov function $W(u,v) = Exp[V(u,v)] - 1$, which is positive definite in a greater region of parameters than the previous one. This allows us to extend the global stability results of reaction diffusion models of Leslie-Gower type to other parameter regions.

For the Leslie-Gower model, if $\alpha = \beta$, the Hessian matrix of the Lyapunov function $V(u,v)$ proposed by Du and Hsu [1] is not definite positive when $u = \frac{\lambda}{\alpha}$, the upper bound of u. However, the Hessian matrix of the Lyapunov function $W(u,v) = Exp[V(u,v)] - 1$ is positive definite and therefore the global stability of the equilibrium is obtained when $\alpha = \beta$, by modifying the previous one. When $\alpha < \beta$, **Theorem 1.3** improves **Theorem 1.1**. It is sufficient that $\dfrac{\mu}{\lambda}$ be large enough to ensure the global stability criterion by the exponential Lyapunov function. It is valid when $\mu \longrightarrow \infty$ or $\lambda \longrightarrow 0$.

For the generalized Leslie-Gower, **Theorem 1.7** improves **Theorems 1.4, 1.5 and 1.6**. We prove that global stability, for $\dfrac{\mu}{\lambda}$ large enough and $\sigma > 1$, is also guaranteed. However, in all works, the results are for different parameter values. The asymptotic behaviors of the parameters μ and λ are the same.

Leslie-Gower models are used in several ecological and ecoepidemiological models, when the disease in the species is considered. Establishing a global stability criterion for the Leslie-Gower type models is important to ensure the co-existence of the species homogeneously in space. There are no non-constant positive steady-states for the reaction diffusion model. In future work, we will explore the development of other Lyapunov functions to extend these results.

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