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Detecting Canard Tori in Quadratic Jerk Systems via Higher-Order Averaging Theory

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Abstract. In this work, we develop a higher–order averaging approach, in the spirit of [1], to detect canard tori in a general quadratic jerk system. By combining classical averaging techniques with methods for the detection of torus bifurcations, we obtain a framework that we apply to a three–dimensional quadratic jerk system exhibiting a zero–Hopf equilibrium. In this context, the loss of stability of a periodic orbit leads to the emergence of a canard torus, which is rigorously detected as a Neimark–Sacker bifurcation in the corresponding Poincaré map. We provide a precise statement of the main result, together with a detailed computational scheme and numerical illustrations.

Keywords. Quadratic Jerk Systems, Invariant Tori, Averaging Theory

1 Introduction

The detection of invariant tori of non-autonomous differential equations, especially when these tori arise from a Neimark–Sacker bifurcation of a periodic orbit, play an importat role in the qualitative analysis of such systems. In some cases, this bifurcation gives rise to a canard phenomenon in the sense described in [2]: trajectories passing through the internal region of the torus (the region with negative curvature, near the z-axis) evolve on a slow time scale, whereas along the external part of the torus (the region with positive curvature) they move much faster, forming a structure that we refer to as a **canard torus**.

In this work, we focus on a class of quadratic jerk systems, that is, third–order differential equations with quadratic nonlinearities, which naturally arise in mechanical and electronic models. In previous work [3], averaging methods were used to detect periodic orbits and to study their bifurcations. Here, we apply new techniques, based on the results of [1], to detect canard tori, which emerge when a periodic solution loses stability as a bifurcation parameter varies.

Our approach is based on a two–parameter family of non-autonomous differential equations written in the standard form

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{k} \varepsilon^i \mathbf{F}_i(t, \mathbf{x}; \mu) + \varepsilon^{k+1} \widetilde{\mathbf{F}}(t, \mathbf{x}; \mu, \varepsilon), \tag{1}$$

where the functions $\mathbf{F}_i$ are T–periodic in time and smooth in their arguments. Following the general methodology of averaging theory, we construct an asymptotic expansion for the solutions and the corresponding Poincaré map, whose first non–vanishing averaged function governs the dynamics near the periodic orbit.

In Section 2, we review the main elements of the averaging approach and its connection with torus bifurcation theory. Section 3 introduces the quadratic jerk system under consideration and discusses the presence of a zero–Hopf equilibrium and its periodic orbit. Our main result on the existence of a canard torus is stated in Section 4 and its proof is done in Section 5. Section 6 is

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devoted to numerical simulations that illustrate the theoretical findings. Finally, in Section 7 we discuss further directions.

2 Preliminaries and Averaging Theory

We consider the non-autonomous system (1) with $\mathbf{x} \in \Omega \subset \mathbb{R}^2$, $\mu \in \mathbb{R}$, and $\varepsilon > 0$ small. The classical averaging method constructs an approximate solution of the form

$$\mathbf{x}(t, \mathbf{z}; \varepsilon) = \mathbf{z} + \varepsilon y_1(t, \mathbf{z}) + \varepsilon^2 y_2(t, \mathbf{z}) + \mathcal{O}(\varepsilon^3),$$

which is valid on time intervals of order $\mathcal{O}(1/\varepsilon)$. As a consequence, the associated Poincaré map (with time–T section) can be written as

$$P(\mathbf{x}; \mu, \varepsilon) = \mathbf{x} + \varepsilon\, \mathbf{g}_1(\mathbf{x}; \mu) + \varepsilon^2\, \mathbf{g}_2(\mathbf{x}; \mu) + \mathcal{O}(\varepsilon^3), \tag{2}$$

where

$$\begin{aligned} \mathbf{g}_1(\mathbf{z}; \mu) &= \int_0^T \mathbf{F}_1(\tau, \mathbf{z}; \mu)\, d\tau, \\ \mathbf{g}_2(\mathbf{z}; \mu) &= \int_0^T \Big[\mathbf{F}_2(\tau, \mathbf{z}; \mu) + \frac{\partial \mathbf{F}_1}{\partial \mathbf{x}}(\tau, \mathbf{z}; \mu) \int_0^\tau \mathbf{F}_1(s, \mathbf{z}; \mu)\, ds\Big] d\tau. \end{aligned}$$

These functions, known as the first and second order averaged functions, determine the bifurcation structure of the fixed point $\boldsymbol{\xi}(\mu, \varepsilon)$ of P.

In particular, assume that $\boldsymbol{\xi}(\mu, \varepsilon)$ is given by

$$P\big(\boldsymbol{\xi}(\mu, \varepsilon); \mu, \varepsilon\big) = \boldsymbol{\xi}(\mu, \varepsilon)$$

and that for $\varepsilon = 0$ one has $\boldsymbol{\xi}(\mu, 0) = \mathbf{x}_\mu$. Suppose further that the Jacobian matrix $D_{\mathbf{x}} P(\boldsymbol{\xi}(\mu, \varepsilon); \mu, \varepsilon)$ has a pair of complex–conjugate eigenvalues

$$\lambda(\mu, \varepsilon) = 1 + \varepsilon\big[\alpha(\mu) \pm i\beta(\mu)\big] + \mathcal{O}(\varepsilon^2),$$

with $\alpha(\mu_0) = 0$, $\beta(\mu_0) = \omega_0 > 0$ for some μ_0. Then, if the first Lyapunov coefficient

$$\ell_1^\varepsilon = \varepsilon\, \ell_{1,1} + \varepsilon^2\, \ell_{1,2} + \mathcal{O}(\varepsilon^3) \tag{3}$$

is nonzero, a Neimark–Sacker (torus) bifurcation occurs. The degree of attraction or repulsion of the invariant circle in the Poincaré map is determined by the order of the first nonzero coefficient of ℓ_1^ε, and we believe this plays a crucial role in the emergence of slow–fast dynamics. Consequently, the bifurcating torus may exhibit canard behavior, meaning that trajectories can follow repelling branches for extended periods, thereby forming a **canard torus**.

The rigorous computation of $\ell_{1,j}$, for $j = 1, 2$, is accomplished by applying a near-identity transformation together with suitable changes of coordinates which, under appropriate non-resonance conditions, reduce the system to its normal form, as discussed in [1].

3 Quadratic Jerk Systems and Canard Phenomena

Consider now the quadratic jerk system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= z, \\ \frac{dz}{dt} &= -a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 - a_5 xy - a_6 xz - a_7 y^2 - a_8 yz - a_9 z^2, \end{aligned} \tag{4}$$

with parameters $a_i \in \mathbb{R}$, $0 \le i \le 9$, and with the assumptions

$$a_4 > 0, \quad \Delta = a_1^2 - 4a_0a_4 > 0.$$

It is straightforward to verify that, under the conditions

$$a_0 = -\frac{a_1^2}{4a_4}, \quad a_3 = -\frac{a_6a_1}{2a_4}, \quad a_2 < \frac{a_1a_5}{2a_4},$$

the point

$$E_0 = \left(-\frac{a_1}{2a_4}, 0, 0\right)$$

is a zero–Hopf equilibrium. Consequently, a periodic orbit emerges from E_0, whose stability may change as parameters vary. In our context, a change in the stability of the periodic orbit is followed by the emergence of an invariant torus, which, due to it slow–fast behaviour in the jerk system, is interpreted as a **canard torus**.

We now introduce a perturbative expansion of the parameters:

$$\begin{aligned} a_0 &= \frac{a_{1,0}^2}{4a_{4,0}} + \varepsilon^2 s_0 & a_3 &= \frac{a_{1,0}a_{6,0}}{2a_{4,0}} + \varepsilon\, s_3, \\ a_1 &= a_{1,0} + \varepsilon^2 s_1 & a_4 &= a_{4,0} + \varepsilon^2 s_4, \\ a_2 &= a_{1,0} + \varepsilon\, s_2 & a_k &= a_{k,0} + \varepsilon\, s_k, \quad 5 \le k \le 9. \end{aligned}$$

In accordance with the calculations presented in [3], the subsequent step involves transforming the system. This modification provide new coefficients to the original system converting(4) into

$$\begin{aligned} \frac{du}{dt} &= -\delta v \\ \frac{dv}{dt} &= \delta u + \alpha_1 u^2 + \alpha_2 vu + \alpha_3 wu + \alpha_4(\eta)\epsilon u + \alpha_5 v^2 + \alpha_6 wv + \alpha_7(\eta)\epsilon v + \alpha_8 w^2 + \alpha_9(\eta)\epsilon w + \cdots \\ \frac{dw}{dt} &= \beta\left[\alpha_1 u^2 + \alpha_2 vu + \alpha_3 wu + \alpha_4(\eta)\epsilon u + \alpha_5 v^2 + \alpha_6 wv + \alpha_7(\eta)\epsilon v + \alpha_8 w^2 + \alpha_9(\eta)\epsilon w\right] + \cdots \end{aligned}$$

where

$$\begin{aligned} \delta &= \sqrt{\frac{a_{5,0}a_{1,0}}{2a_{4,0}} - a_{2,0}} & \alpha_4(\eta) &= -\frac{a_{5,0}\eta - s_5a_{1,0} + 2s_2a_{4,0}}{\xi} & \alpha_8 &= a_{4,0} \\ \alpha_1 &= \frac{4a_{7,0}a_{4,0}^2}{\xi^2} & \alpha_5 &= \frac{a_{9,0}\xi^4 - 4a_{6,0}a_{4,0}^2\xi^2 + 16a_{4,0}^5}{\xi^4} & \alpha_9(\eta) &= \eta \\ \alpha_2 &= -\frac{2a_{4,0}\left(\xi^2 a_{8,0} - 4a_{4,0}^2 a_{5,0}\right)}{\xi^3} & \alpha_6 &= \frac{a_{6,0}\xi^2 - 8a_{4,0}^3}{\xi^2} & & \\ \alpha_3 &= -2\frac{a_{5,0}a_{4,0}}{\xi} & \alpha_7(\eta) &= \frac{a_{6,0}\eta\xi^2 - 8\eta a_{4,0}^3 - s_6a_{1,0}\xi^2 + 2s_3\xi^2 a_{4,0}}{2\xi^2 a_{4,0}} & & \end{aligned}$$

and (see [3]) we dropped terms that are not used in the following analysis.

Under these assumptions, the first order averaging method can be used to show that system (4) possesses a unique limit cycle

$$\varphi(t;\alpha_9,\varepsilon) = (x(t;\alpha_9,\varepsilon), y(t;\alpha_9,\varepsilon), z(t;\alpha_9,\varepsilon))$$

for $\varepsilon > 0$ small emerging from the origin of coordinates.

4 Main Result: Existence of a Canard Torus

The following theorem, adapted from Theorem B in [1], provides sufficient conditions for the emergence of a torus via a Neimark–Sacker bifurcation in the averaged Poincaré map. The original theorem is formulated in greater generality, taking into consideration the averaged function of arbitrary order. Here, we present a simpler, but still robust, formulation, intended to be accessible to readers of all backgrounds and easy to apply to their own problems. This theorem relies only on the first and second averaged functions; nevertheless most applications of this results derive from second-order analysis.

Theorem 4.1. *Let $l \in \{1, 2\}$ be the index of the first non–vanishing averaged function and assume that the Poincaré map associated to the periodic orbit $\Phi(t; \mu, \varepsilon)$ of system (4) can be written as*

$$P(\mathbf{x}; \mu, \varepsilon) = \mathbf{x} + \varepsilon\, \mathbf{g}_1(\mathbf{x}; \mu) + \varepsilon^2\, \mathbf{g}_2(\mathbf{x}; \mu) + \mathcal{O}(\varepsilon^3),$$

with fixed point $\boldsymbol{\xi}(\mu, \varepsilon)$ such that $\boldsymbol{\xi}(\mu, 0) = \mathbf{x}_\mu$. Suppose that:

(B1) *The averaged functions satisfy $\mathbf{g}_l(\mathbf{x}_\mu; \mu) = 0$ for every μ in a small interval J_0 and the eigenvalues of $D_{\mathbf{x}}\mathbf{g}_l(\mathbf{x}_\mu; \mu)$ are given by $\alpha(\mu) \pm i\beta(\mu)$ with $\alpha(\mu_0) = 0$ and $\beta(\mu_0) = \omega_0 > 0$.*

(B2) *The transversality condition*

$$\left.\frac{d\alpha(\mu)}{d\mu}\right|_{\mu=\mu_0} = d \neq 0,$$

holds.

(B3) *For each $\varepsilon \in (0, \varepsilon_1)$ the Jacobian $D_{\mathbf{x}}P(\boldsymbol{\xi}(\mu, \varepsilon); \mu, \varepsilon)$ has the expansion*

$$D_{\mathbf{x}}P(\boldsymbol{\xi}(\mu, \varepsilon); \mu, \varepsilon) = I + \varepsilon A_1 + \varepsilon^2 A_2 + \mathcal{O}(\varepsilon^3),$$

with A_1 and A_2 in real Jordan form.

If the first Lyapunov coefficient

$$\ell_1^\varepsilon = \varepsilon\, \ell_{1,1} + \varepsilon^2\, \ell_{1,2} + \mathcal{O}(\varepsilon^3)$$

satisfies $\ell_{1,j^*} \neq 0$ for some $j^* \in \{l, 2\}$, then there exists, for each $\varepsilon > 0$ sufficiently small, a C^1 curve $\mu(\varepsilon)$ with $\mu(0) = \mu_0$, and neighborhoods $\mathcal{U}_\varepsilon \subset \mathbb{S}^1 \times \Omega$ and $J_\varepsilon \subset J_0$, such that:

1. *For $\mu \in J_\varepsilon$ with $\ell_{1,j^*}(\mu - \mu(\varepsilon)) \geq 0$, the periodic orbit $\Phi(t; \mu(\varepsilon), \varepsilon)$ is (un)stable (depending on the sign of ℓ_{1,j^*}) and no invariant torus exists in $\mathcal{U}_\varepsilon$.*
2. *For $\mu \in J_\varepsilon$ with $\ell_{1,j^*}(\mu - \mu(\varepsilon)) < 0$, the system (4) admits a unique invariant torus $T_{\mu,\varepsilon} \subset \mathcal{U}_\varepsilon$ surrounding $\Phi(t; \mu, \varepsilon)$. Moreover, if $\ell_{1,j^*} > 0$ (respectively, $\ell_{1,j^*} < 0$), then $T_{\mu,\varepsilon}$ is unstable (respectively, stable) and the periodic orbit is stable (respectively, unstable).*

In the slow–fast regime, such an invariant torus, which bifurcates from a periodic orbit undergoing a change in stability, is identified as a ***canard torus****.*

In addition, by performing an explicit computation of $\ell_{1,2}$ and a careful change of coordinates (see the detailed computations in [1]), one obtains the expressions for the Lyapunov coefficient

$$\begin{aligned}\ell_{1,2} = {}& \beta\delta^2(\alpha_1 + \alpha_5)\big(48\alpha_8^2\beta^3(-7\alpha_1\alpha_3 + 4\pi\alpha_1\alpha_6 + \alpha_2\alpha_6 - 9\alpha_3\alpha_5 + 4\pi\alpha_5\alpha_6) \\ & + 16\alpha_8\beta^2(6\alpha_2\alpha_8(\alpha_1 + \alpha_5) - \alpha_6(3\alpha_3(\alpha_1 + 2\alpha_5) + 18\pi\alpha_6(\alpha_1 + \alpha_5) - 2\alpha_2\alpha_6)) \\ & + \alpha_6\beta(\alpha_6(-7\alpha_1\alpha_3 + 12\pi\alpha_1\alpha_6 + \alpha_2\alpha_6 - 5\alpha_3\alpha_5 + 12\pi\alpha_5\alpha_6) + 64\alpha_2\alpha_8(\alpha_1 + \alpha_5)) \\ & - 6\alpha_2\alpha_6^2(\alpha_1 + \alpha_5)\big),\end{aligned}$$

and bifurcation parameter curve

$$\mu(\varepsilon) = \frac{2\alpha_7\alpha_8}{\alpha_6} + \epsilon\frac{\alpha_7^2\alpha_8}{4\alpha_6^3\beta\delta(\alpha_1+\alpha_5)}\left(2\alpha_8\beta^2(\alpha_1\alpha_3+\alpha_2\alpha_6-\alpha_3\alpha_5)\right. \\ + \alpha_6\beta(\alpha_1(\alpha_3-8\pi\alpha_6)+\alpha_2\alpha_6-\alpha_5(\alpha_3+8\pi\alpha_6)) + 2\alpha_2\alpha_6(\alpha_1+\alpha_5) \\ \left. + 4\alpha_2\alpha_8\beta(\alpha_1+\alpha_5)\right) + \mathcal{O}(\varepsilon^2), \tag{5}$$

where Δ_1 is given in terms of the system parameters. The sign of $\ell_{1,2}(\alpha_9 - \mu(\varepsilon))$ then determines the existence and stability of the canard torus.

5 Proofs and Computational Scheme

We briefly outline the steps used to derive the result of Theorem 4.1 for system (4). The aim of this section is to provide the reader with a step-by-step guide to obtaining similar results in their own problems. For a complete proof of Theorem 4.1 in a more general setting, we recommend [1].

Step 1. Reduction and Averaging: A reduction of system (4) to a planar form is achieved by first performing a suitable coordinate transformation that isolates the center manifold corresponding to the zero–Hopf equilibrium. The reduced system is expressed in the polar coordinates (R, W) and, after rescaling time (with θ as the new independent variable), one obtains a system of the form

$$\frac{dR}{d\theta} = \varepsilon\, F_1(\theta, R, W; \alpha_9) + \mathcal{O}(\varepsilon^2), \quad \frac{dW}{d\theta} = \varepsilon\, F_2(\theta, R, W; \alpha_9) + \mathcal{O}(\varepsilon^2).$$

The associated Poincaré map, computed at time 2π, then takes the form (2) with averaged functions

$$\mathbf{g}_1(\mathbf{x}; \alpha_9) = 2\pi\,(f_1(R, W), f_2(R, W)),$$

and similarly for $\mathbf{g}_2$.

Step 2. Fixed Point and Linearization: The fixed point $\boldsymbol{\xi}(\alpha_9, \varepsilon)$ of the Poincaré map is expanded as

$$\boldsymbol{\xi}(\alpha_9, \varepsilon) = (R^*, W^*) + \varepsilon\,(R_1, W_1) + \mathcal{O}(\varepsilon^2),$$

where (R^*, W^*) is determined by the equation $\mathbf{g}_1(R^*, W^*; \alpha_9) = 0$. The linearization around this fixed point yields eigenvalues that, by hypothesis, cross the unit circle when μ passes through $\mu(\varepsilon)$.

Step 3. Lyapunov Coefficient Computation: Using the near–identity transformation (see, e.g., [1]), the first Lyapunov coefficient ℓ_1^ε of the Poincaré map is computed. In the quadratic jerk system, one finds that

$$\ell_1^\varepsilon = \varepsilon^2\, \ell_{1,2} + \mathcal{O}(\varepsilon^3),$$

where $\ell_{1,2}$ is an explicitly determined function of the parameters α_i, β, and δ. The sign of $\ell_{1,2}$ is crucial for the subsequent bifurcation analysis.

Step 4. Verification of Hypotheses and Canard Behavior: Under hypotheses (B1)–(B3) and the additional nondegeneracy condition on $\ell_{1,2}$, one applies the Implicit Function Theorem to obtain the bifurcation curve (5). When $\alpha_9 - \mu(\varepsilon)$ changes sign, the periodic orbit loses stability and a canard torus emerges, capturing trajectories that follow both attracting and repelling regions.

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Step 5. Conclusion of the Proof: The final step is to apply a version of the Neimark–Sacker bifurcation theorem (as stated in Theorem 4.1) to conclude the existence, uniqueness, and stability properties of the bifurcating torus. Detailed computations of the change–of–coordinate matrices and the averaged functions guarantee that the hypotheses are met, thus completing the proof.

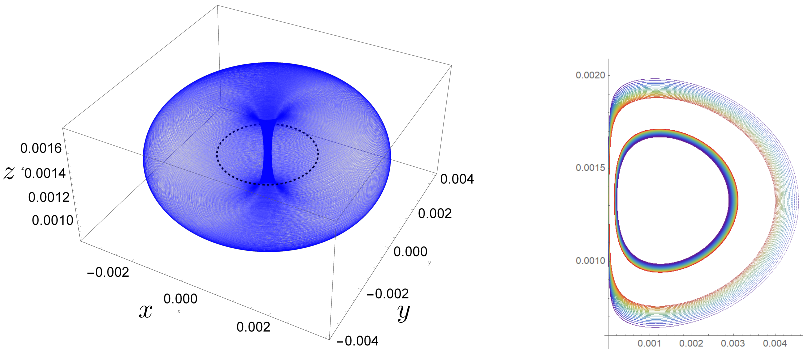
6 Numerical Illustrations

To illustrate the theoretical results, we consider the following set of parameter values:

$$\alpha_1 = 1, \quad \alpha_2 = -1, \quad \alpha_3 = \frac{397}{4}, \quad \alpha_4 = -1, \quad \alpha_5 = 1, \quad \alpha_6 = -1,$$

$$\alpha_7 = 1, \quad \alpha_8 = 1, \quad \alpha_9 = \frac{512\pi - 174705}{84000}, \quad \beta = \frac{1}{64}, \quad \delta = \frac{7}{16}, \quad \varepsilon = \frac{1}{750}.$$

Under these choices, numerical integration of system (4) shows that a limit cycle $\varphi(t; \mu(\varepsilon), \varepsilon)$ is born at the zero–Hopf equilibrium, and as the parameter α_9 is varied through $\mu(\varepsilon)$, an invariant torus emerges. Figure 1 displays (a) a three–dimensional plot of the canard torus (solid line) along with the underlying limit cycle (dashed line) and (b) the corresponding Poincaré section.



(a) (b)

Figura 1: (a) Canard torus (solid) and limit cycle (dashed) in system (4). (b) Poincaré section $y = 0$, $x > 0$ showing the attracting invariant closed curve corresponding to the canard torus. Fonte: elaborado pelo autor.

The numerical results confirm that, for $\alpha_9 - \mu(\varepsilon) < 0$ (or > 0, depending on the sign of $\ell_{1,2}$), the periodic orbit is either stable or unstable, while the canard torus exhibits the complementary stability property, in agreement with Theorem 4.1.

7 Discussion and Conclusions

In this paper we have applied the extended the averaging methodology developed in [1] to detect a canard torus bifurcation in a quadratic jerk system. Our main contribution is the rigorous identification of the parameter regime in which a periodic orbit loses stability via a Neimark–Sacker bifurcation and a canard torus is born. The analysis required the computation of the averaged Poincaré map up to second order, the verification of nonresonance conditions, and a careful calculation of the first Lyapunov coefficient.

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The detection of canard phenomena in three dimensional systems is of considerable interest, and as far as we know this is the first analycal detection of this kind of phenomena for in quadratic jerk systems. Our result also open the door to further investigaions into higher–dimensional and more complex models.

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