PARABOLIC EQUATIONS IN A SINGULARLY TIME DEPENDENT DOMAIN

Jamil V. Pereira*, Ricardo P. Silva†

*UNESP - Universidade Estadual Paulista "Júlio de Mesquita Filho", Instituto de Geociências e Ciências Exatas,
Departamento de Matemática, Rio Claro SP, Brazil

†UNESP - Universidade Estadual Paulista "Júlio de Mesquita Filho", Instituto de Geociências e Ciências Exatas,
Departamento de Matemática, Rio Claro SP, Brazil

Emails: jamil@rc.unesp.br, rpsilva@rc.unesp.br

Abstract— We study the limiting regime of nonlinear parabolic equations posed in a time dependent family of domains $\{\Omega_t^\epsilon\}_{t\in\mathbb{R}} \subset \mathbb{R}^{n+1}$ which collapses to a lower dimensional set as the parameter ϵ goes to 0.

Keywords— evolution process, thin domains, noncylindrical domain

1 Introduction

In this work, based on paper [5], we are concerned with the effective behavior of a nonlinear reaction-diffusion equation posed in a time dependent family of domains which collapses to a lower dimensional set. Inspired by the recent works [3,4], both of them related with asymptotic behavior of reaction-diffusion equations on time-dependent domains, we are interested with reaction-diffusion equations in a *time dependent thin domain*. For the best of our knowledge this is an untouched topic in the literature.

In order to set up the problem, let ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, and $g \in C^2(\overline{\omega}\times\mathbb{R};\mathbb{R})$ satisfying

(G_1) There exist positive constants α_1 and α_2 such that

$$\alpha_1 \leq g,\ |\nabla_x g|,\ g_t,\ g_{tt} \leq \alpha_2,$$

for all $(x,t) \in \omega\times\mathbb{R}$;

(G_2) There exists a constant k such that

$$|g_{x_i}(x,t) - g_{x_i}(x,s)| \leq k|t-s|,$$

for all $(x,t)\in\omega\times\mathbb{R}$, $i = 1,2,\cdots,n$.

In the following, ϵ will denote a positive parameter which will converge to zero. Fixed $t\in\mathbb{R}$ we define the time-dependent thin domain

$$\Omega_t^\epsilon := \{(x,y)\in\mathbb{R}^{n+1} : x\in\omega,\ 0<y<\epsilon g(x,t)\}.$$

For each $\tau\in\mathbb{R}$ and $\epsilon \geq 0$ we set the domain

$$Q_\tau^\epsilon := \bigcup_{t\in(\tau,\infty)} \Omega_t^\epsilon \times \{t\},$$

as well the lateral boundary

$$\Sigma_\tau^\epsilon := \bigcup_{t\in(\tau,\infty)} \partial\Omega_t^\epsilon \times \{t\},$$

where $\Omega_t^0 := \omega$ for all $t\in(\tau,\infty)$.

For positive values of the parameter ϵ, we consider the semilinear reaction-diffusion equation

$$\begin{cases} u_t^\epsilon - \Delta u^\epsilon + u^\epsilon = f(u^\epsilon), & \text{in } Q_\tau^\epsilon, \\ \dfrac{\partial u^\epsilon}{\partial \eta_\tau^\epsilon} = 0, & \text{on } \Sigma_\tau^\epsilon, \\ u^\epsilon(\cdot,\tau) = u_\tau^\epsilon, & \text{in } \Omega_\tau^\epsilon, \end{cases} \tag{1}$$

where η_τ^ϵ denotes the unit outward normal vector field to Σ_τ^ϵ, $\frac{\partial}{\partial\eta_\tau^\epsilon}$ denotes the outwards normal derivative and $f:\mathbb{R}\to\mathbb{R}$ is a C^2-function with bounded derivatives up second order.

Besides, since our interest resides in the asymptotic behavior of the solutions and its dependence with respect to ϵ, we will require that solutions of (1) are bounded for large values of time. A natural assumption to obtain this boundedness is expressed in the following dissipative condition

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < 0. \tag{2}$$

This implies for any $\eta > 0$, the existence of a positive constant c_η such that,

$$f(s)s \leq \eta s^2 + c_\eta, \quad \forall\ s\in\mathbb{R}.$$

In the analysis of the limiting behavior of the problem (1), it will be useful to introduce the domain $\Omega := \omega\times(0,1)$, independent of ϵ and t, which is obtained from Ω_t^ϵ by the change of coordinates

$$\begin{aligned} \mathcal{T}_t^\epsilon : \Omega &\to \Omega_t^\epsilon \\ (x,y) &\mapsto (x, \epsilon\, g(x,t)y) \end{aligned}$$

Such change of coordinates induces an isomorphism from $W^{m,p}(\Omega_t^\epsilon)$ onto $W^{m,p}(\Omega)$ by

$$u \overset{\Phi_t^\epsilon}{\longmapsto} v := u\circ\mathcal{T}_t^\epsilon \tag{3}$$

with partial derivatives related by

$$\begin{aligned} u_t &= v_t - \frac{yg_t}{g} v_y \\ u_{x_i} &= v_{x_i} - \frac{yg_{x_i}}{g} v_y, \quad i = 1, \dots, n \\ u_y &= \frac{1}{\epsilon\, g} v_y. \end{aligned} \tag{4}$$

In this new coordinates we rewrite equation (1) as the following (nonautonomous) equation posed in the fixed domain Ω,

$$\begin{cases} v_t^\epsilon - \dfrac{1}{g}\mathrm{div} B_\epsilon(t) v^\epsilon - y\dfrac{g_t}{g} v_y^\epsilon + v^\epsilon = f(v^\epsilon), & \text{in } \Omega, \\ B_\epsilon(t) v^\epsilon \cdot \eta = 0, & \text{on } \partial\Omega, \\ v^\epsilon(\cdot, \tau) = u_\tau^\epsilon \circ \mathcal{T}_\tau^\epsilon, & \text{in } \Omega, \end{cases} \tag{5}$$

where η denotes the unit outward normal vector field to $\partial\Omega$, and

$$B_\epsilon(t)v = \begin{bmatrix} g v_{x_1} - y g_{x_1} v_y \\ \vdots \\ g v_{x_n} - y g_{x_n} v_y \\ -\displaystyle\sum_{i=1}^n y g_{x_i} v_{x_i} + \frac{1}{\epsilon^2 g}\Big(1 + \sum_{i=1}^n (\epsilon y g_{x_i})^2\Big) v_y \end{bmatrix}$$

After a careful study of the solutions of (5), one starts to suspect that v^ϵ tends not to depend on the variable y as $\epsilon \to 0$. Therefore, if a limiting regime for the problem (1) exists, then it should be given by the (non-autonomous) problem

$$\begin{cases} v_t - \dfrac{1}{g}\displaystyle\sum_{i=1}^n (g v_{x_i})_{x_i} + v = f(v), & \text{in } \omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & \text{on } \partial\omega, \\ v(\cdot, \tau) = v_\tau, & \text{in } \omega, \end{cases} \tag{6}$$

where ν denotes the unit outward normal vector field to $\partial\omega$.

The comparison between solution of the problems (5) and (6) is the aim of this paper.

2 Abstract Formulation

We start stressing the fact that Ω_t^ϵ varies in accordance with a positive parameter ϵ, collapsing themselves to the lower dimensional domain ω as ϵ goes to 0. Therefore, in order to preserve the "relative capacity" of a mensurable subset $E \subset \Omega_t^\epsilon$, we rescale the Lebesgue measure by a factor $1/\epsilon$ dealing with the singular measure $\rho^\epsilon(E) = \epsilon^{-1}|E|$.

As we will see, it will be convenient to consider the space $H_\epsilon := H^1(\Omega)$ endowed with the equivalent norm

$$\|v\|_{H_\epsilon} := \left[\int_\Omega \left(|\nabla_x v|^2 + \frac{1}{\epsilon^2}|v_y|^2 + |v|^2\right) dxdy\right]^{\frac{1}{2}}.$$

It is immediate consequence of (G_1) that the family of isomorphism $\{\Phi_t^\epsilon\}$ satisfies

$$\|\Phi_t^\epsilon\|_{\mathcal{L}(H^1(\Omega_t^\epsilon;\rho^\epsilon), H_\epsilon)} \le c,$$

for some positive constant c independent on ϵ and t.

For each pair of parameters $(\epsilon, t) \in (0, 1] \times \mathbb{R}$, we consider the sesquilinear form

$$\begin{aligned} &a_t^\epsilon : H_\epsilon \times H_\epsilon \to \mathbb{R} \\ a_t^\epsilon(u, v) = &\int_\Omega (B_\epsilon(t)u \cdot \nabla v - y g_t(x,t) u_y v \\ &+ g(x,t) uv)\, dxdy. \end{aligned} \tag{7}$$

As a first remark, we notice that under assumptions (G_1), a_t^ϵ is a continuous form and there exist positive constants c_1, c_2, independents on ϵ and t, such that

$$c_1\|v\|_{H_\epsilon}^2 \le a_t^\epsilon(v, v) \le c_2\|v\|_{H_\epsilon}^2, \tag{8}$$

for all $v \in H_\epsilon$ and $(\epsilon, t) \in (0, \bar\epsilon] \times \mathbb{R}$.

Since H_ϵ is densely and compactly embedding in $L^2(\Omega)$, the sesquilinear form a_t^ϵ yields a densely defined positive linear operator with compact resolvent, $A_\epsilon(t) : D(A_\epsilon(t)) \subset L^2(\Omega) \to L^2(\Omega)$, which is defined by the relation

$$a_t^\epsilon(u, v) = (A_\epsilon(t)u, v)_t, \quad u \in H_\epsilon,\ v \in D(A_\epsilon(t)),$$

where $(u, v)_t := \displaystyle\int_\Omega g(x,t) uv\, dxdy.$

By regularity of $\partial\omega$ we notice that

$$D(A_\epsilon(t)) = \{v \in H^2(\Omega) : B_\epsilon(t)v \cdot \eta = 0\},$$

is independent on ϵ. Moreover

$$A_\epsilon(t)v = -\frac{1}{g(\cdot,t)}\mathrm{div} B_\epsilon(t)v - y\frac{g_t(\cdot,t)}{g(\cdot,t)} v_y + v,$$

for all $v \in D(A_\epsilon(t))$.

Multiplying equation (5) by $\varphi \in H^1(\Omega)$ and integrating by parts we get

$$(v_t^\epsilon, \varphi)_t + a_t^\epsilon(v^\epsilon, \varphi) = (f(v^\epsilon), \varphi)_t.$$

Hence we can write equation (5) as an abstract equation

$$\frac{dv^\epsilon}{dt}(t) + A_\epsilon(t)v^\epsilon(t) = f^e(v^\epsilon(t)), \tag{9}$$

where f^e is the Nemitskii operator (composition operator) associated to f.

Combining assumptions (G_1)-(G_2), we also have that

$$|a_t^\epsilon(u, v) - a_s^\epsilon(u, v)| \le k_1|t - s|\|u\|_{H_\epsilon}\|v\|_{H_\epsilon}, \tag{10}$$

for some constant k_1 independent on ϵ, $t, s \in \mathbb{R}$, and $u, v \in H_\epsilon$. Therefore, thanks to Theorem 5.4.2 in [6] there exists an unique solution of the linear homogeneous problem

$$\begin{cases} \dfrac{dv^\epsilon}{dt}(t) + A_\epsilon(t)v^\epsilon(t) = 0, & t \ge \tau \in \mathbb{R} \\ v^\epsilon(\tau) = v_\tau^\epsilon \in H_\epsilon. \end{cases} \tag{11}$$

This allow us to consider for each value of the parameter ϵ, each initial time $\tau \in \mathbb{R}$ and each initial data $v_\tau^\epsilon \in H_\epsilon$, the solution $v^\epsilon(\cdot,\tau,v_\tau^\epsilon) \in C^1([\tau,\infty);H_\epsilon)$ of (11). This give raise a linear process $\{L_\epsilon(t,\tau), t \geq \tau\} \subset \mathcal{L}(H_\epsilon)$ defined by $L_\epsilon(t,\tau)v_\tau^\epsilon := v^\epsilon(t,\tau,v_\tau^\epsilon)$. We notice that (11) is the abstract Cauchy problem associated to the equation (5) in the case $f \equiv 0$.

Since we are assuming the nonlinearity $f \in C^2(\mathbb{R};\mathbb{R})$ bounded as well as its derivatives up second order, local existence of the nonlinear counterpart is guaranteed by Theorem 6.6.1 in [6], i.e, writing the problem (5) as

$$\begin{cases} \dfrac{dv^\epsilon}{dt}(t) + A_\epsilon(t)v^\epsilon(t) = f^e(v^\epsilon(t)), \\ v^\epsilon(\tau) = v_\tau^\epsilon \in H_\epsilon, \end{cases} \tag{12}$$

there exist a time $T_\tau > 0$ and an unique solution $v^\epsilon(\cdot,\tau,v_\tau^\epsilon) \in C^1([\tau,\tau+T_\tau];H_\epsilon)$ of (12). Under dissipative assumption (2) on the nonlinearity f, one can show that actually $v^\epsilon(\cdot,\tau,v_\tau^\epsilon) \in C^1([\tau,\infty);H_\epsilon)$. Further details can be found in [1] and [2].

Similarly to the linear case, this allow us to consider for each value of the parameter ϵ, each initial time $\tau \in \mathbb{R}$, and each initial data $v^\epsilon \in H_\epsilon$, the (nonlinear) evolution process $\{S_\epsilon(t,\tau) : t \geq \tau\}$ in the state space H_ϵ defined by $S_\epsilon(t,\tau)v^\epsilon := v^\epsilon(t,\tau,v^\epsilon)$.

By reader's convenience we recall the definition of an evolution process in a Banach space

Definition 1 *We say that a family of maps $\{S(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ from a Banach space $\mathcal{X}$ into itself is an evolution process if*

(i) $S(\tau,\tau) = I$ *(identity operator in $\mathcal{X}$), for any $\tau \in \mathbb{R}$,*

(ii) $S(t,\sigma)S(\sigma,\tau) = S(t,\tau)$*, for any $t \geqslant \sigma \geqslant \tau$,*

(iii) $(t,\tau) \mapsto S(t,\tau)v$ *is continuous for all $t \geqslant \tau$ and $v \in \mathcal{X}$.*

3 Limiting consideration

For each $t \in \mathbb{R}$ we consider the sesquilinear form

$$a_t^0 : H^1(\omega) \times H^1(\omega) \to \mathbb{R}$$

defined by

$$a_t^0(u,v) = \int_\omega g(x,t)\big(\nabla u \cdot \nabla v + uv\big)dx.$$

With this definition we immediate have that

$$\alpha_1 \|v\|^2_{H^1(\omega)} \leq a_t^0(v,v) \leq \alpha_2 \|v\|^2_{H^1(\omega)}, \tag{13}$$

for all $v \in H^1(\omega)$.

Similarly, a_t^0 gives rise a densely defined positive linear operator with compact resolvent, $A_0(t) : D(A_0(t)) \subset L^2(\omega) \to L^2(\omega)$, defined by relation

$$a_t^0(u,v) = ((A_0(t)u,v))_t,\ u \in H^1(\omega),\ v \in D(A_0(t))$$

where $((u,v))_t := \displaystyle\int_\omega g(x,t)uv\,dxdy,\ u,v \in L^2(\omega)$.

By regularity of $\partial\omega$,

$$D(A_0(t)) = \{v \in H^2(\omega) : \nabla v \cdot \eta = 0\},$$

and is independent on t. Moreover

$$A_0(t)v = -\frac{1}{g(\cdot,t)}\sum_{i=1}^n (g(\cdot,t)v_{x_i})_{x_i} + v,\ v \in D(A_0(t)).$$

By (G_1)-(G_2) there exists a constant k_2 (independent on t) such that

$$|a_t^0(u,v) - a_t^0(u,v)| \leq k_2|t-s|\|u\|_{H^1(\omega)}\|v\|_{H^1(\omega)},$$

for all $t,s \in \mathbb{R}$ and $u,v \in H^1(\omega)$.

Therefore, writing equation (6) as an abstract evolution equation

$$\begin{cases} \dfrac{dv^0}{dt}(t) + A_0(t)v^0(t) = f^e(v^0(t)) \\ v^0(\tau) = v_\tau^0 \in H^1(\omega), \end{cases} \tag{14}$$

we can define a (nonlinear) evolution process $\{S_0(t,\tau) : t \geq \tau\}$ in the state space $H^1(\omega)$ by $S_0(t,\tau)v^0 := v^0(t,\tau,v^0)$

Now we have now the elements to state our results

Lemma 2 *If $\{f^\epsilon\}$ is a bounded family in $L^2(\Omega)$ then $\{A_\epsilon(t)^{-1}f^\epsilon\}$ is a bounded family in H_ϵ.*

Lemma 3 *Let $\{f^\epsilon\}$ be a bounded family in $L^2(\Omega)$. If $Mf^\epsilon \rightharpoonup \hat{f}$ w-$L^2(\omega)$, then*

$$\|A_\epsilon(t)^{-1}f^\epsilon - EA_0(t)^{-1}\hat{f}\|_{H_\epsilon} \overset{\epsilon\to 0}{\longrightarrow} 0, \tag{15}$$

uniformly in t in bounded subsets of $\mathbb{R}$, where the extension operator E is defined as

$$\begin{aligned} E : H^1(\omega) &\to H^1(\Omega) \\ (Eu)(x,y) &= u(x) \end{aligned} \tag{16}$$

Theorem 4 *Under the assumptions $(G1)$, $(G2)$ on the profile $g \in C^2(\overline{\omega} \times \mathbb{R};\mathbb{R})$, and assuming that the nonlinearity $f \in C^2(\mathbb{R};\mathbb{R})$ is bounded with bounded derivatives up second order, then equations (12) and (14) generate evolution processes $\{S_\epsilon(t,\tau) : t \geq \tau\}$ and $\{S_0(t,\tau) : t \geq \tau\}$ in H_ϵ and $H^1(\omega)$ respectively. Moreover, given $v^\epsilon \in H_\epsilon$ and $v^0 \in H^1(\omega)$ such that $v^\epsilon \overset{\epsilon\to 0}{\longrightarrow} Ev^0$ in $L^2(\Omega)$, then*

$$\|S_\epsilon(t,\tau)v^\epsilon - ES_0(t,\tau)v^0\|_{H_\epsilon} \overset{\epsilon\to 0}{\longrightarrow} 0,$$

uniformly for (t,τ), $t \geq \tau$, in bounded subsets of $\mathbb{R}^2$.

Proof: *See* [5]. □

DOI: 10.5540/03.2013.001.01.0052

Acknowledgment

RPS is partially supported by FAPESP #2012/06753-8 and FUNDUNESP #0135812, Brazil.

References

[1] V.L. Carbone, M.J.D. Nascimento, K. Schiabel-Silva, R.P. Silva, *Pullback attractors for a singularly nonautonomous plate equation*, Electr. J. Differential Equations, 77, 1–13, (2011).

[2] J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, 25, AMS, (1988).

[3] P.E. Kloeden, P. Marín-Rubio, J. Real, *Pullback attractors for a semilinear heat equation in a non-cylindrical domain*, J. Differential Equations, 244, 2062–2091, (2008). DOI: 10.1016/j.jde.2007.10.031

[4] P.E. Kloeden, J. Real, C. Sun, *Pullback attractors for a semilinear heat equation on time-varying domains*, J. Differential Equations, 246, 4702–4730, (2009). DOI: 10.1016/j.jde.2008.11.017

[5] J.V. Pereira, R.P. Silva, *Reaction-diffusion equations in a noncylindrical thin domain*, submitted, (2013). http://www.rc.unesp.br/igce/matematica/rpsilva/pre-prints/pre-prints/Pre-print.html

[6] H. Tanabe, *Equations of Evolution*, Pitman Publishing Limited, (1979).

