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Fourier Series defined in a Hilbert Space of Fuzzy Numbers

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Abstract. This study conducts an introductory investigation of the Fourier series with fuzzy coefficients. A structure of Hilbert space is provided throughout an identification of the convergent square summable real sequences into a set generated by an enumerable family of strongly linearly independent subsets of fuzzy numbers. An application to a signal processing model with the fuzzy Fourier series approximation of the square wave is presented.

Keywords. Space of Square Summable Real Sequences, Strong Linear Independence, Hilbert Space of Fuzzy Numbers, Fuzzy Fourier Series

1 Introduction

Zadeh's extension sum and scalar multiplication defined in the set of fuzzy numbers lack critical features; for instance, the sum operation does not have an additive inverse, and the multiplication by a scalar verifies the distributive law of a fuzzy set over a sum of scalars only for nonnegative scalar products. Furthermore, while the Pompeiu-Hausdorff distance endows the set of fuzzy numbers with the structure of a complete metric space, this space is nonseparable. Various researchers have proposed solutions to these limitations. Fourier Analysis is a well-known field in classical theory [2, 6]. However, its role in fuzzy literature is still limited. Kadak and Basar studied the Fourier series for approximating fuzzy number-valued functions in [8].

Recent works by Esmi et al. and Laiate have shown that finite Banach spaces of fuzzy numbers emerge via an isomorphism between $\mathbb{R}^n$ and linear combinations of Strongly Linearly Independent (SLI) sets [3, 10]. In recent research [11] this methodology has been extended to a broader set of fuzzy numbers through the construction of a bijection ψ between a subspace of $\ell^p(\mathbb{R})$, $p \in [1, \infty)$, the space of p summable real sequences [13]), and the set generated by an enumerable family of SLI fuzzy numbers. This bijection induces both addition and scalar multiplication operations that provide a vector space structure to the set of fuzzy numbers. Moreover, the function ψ identifies a subspace of $\ell^p(\mathbb{R})$ with a subset of $\mathbb{R}_{\mathcal{F}}$. The completeness of the subspace of $\ell^p(\mathbb{R})$ ensures that the embedded set inherits the structure of an infinite dimensional Banach space for $p \neq 2$ and Hilbert space for $p = 2$. The enumerability of the space comes from the existence of a Schauder basis. Based on this research, an application for the Fourier series of fuzzy numbers is presented in this study, representing a promising direction for applications where fuzzy uncertainty is presented.

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2 Fuzzy Numbers, Zadeh Extension and Strongly Linearly Independent sets

A fuzzy subset A is the graph of its membership function $\mu_A : X \to [0,1]$, where the set X is the universe of discourse. Denoting $\mu_A(x) = A(x)$, then the fuzzy set is given by $A(x) = \{(x, A(x)), x \in X\}$. The α-level of the fuzzy set A is given by $[A]_\alpha = \{x \in U, A(x) \geq \alpha\}$, for each $\alpha \in (0,1]$ and, for $\alpha = 0$, it is defined as $[A]_0 = \overline{\{x \in X, A(x) > 0\}}$, where $\overline{U}$ is the topological closure of set U.

A fuzzy number is a fuzzy set that verifies $X = \mathbb{R}$, and all its α-levels are nonempty and bounded sets of $\mathbb{R}$. In particular, any real number r can be identified with a fuzzy number R whose membership function is given by $R(x) = 1$ if $x = r$ and $R(x) = 0$ otherwise, called a singleton.

An important property of fuzzy numbers is that their α-levels can be represented by closed intervals of $\mathbb{R}$. Indeed, if A is a fuzzy number then $[A]_\alpha = [L_A(\alpha), U_A(\alpha)]$, where $L_A(\alpha)$ and $U_A(\alpha)$ are the endpoints of $[A]_\alpha$, for all $\alpha \in [0,1]$. The class of all fuzzy numbers is denoted by $\mathbb{R}_\mathcal{F}$. The function is then defined as $\delta_A : [0,1] \to \mathbb{R}$ given by $\delta_A(\alpha) = L_A(\alpha) + U_A(\alpha)$, for all $\alpha \in [0,1]$. The function δ_A belongs to the set of all functions $f : [0,1] \to \mathbb{R}$ bounded and equipped with the norm $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$, denoted by $\mathcal{B}([0,1], \mathbb{R})$. The Pompeiu-Hausdorff distance between the fuzzy numbers A and B is given by $\mathcal{D}_\infty(A,B) = \sup_{\alpha \in [0,1]} \max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}$, where $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$ and $[B]_\alpha = [b_\alpha^-, b_\alpha^+]$ for all $\alpha \in [0,1]$.

Let X and Y be two universes of discourse, f a function from X to Y. The function $\widehat{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is called the Zadeh extension of f if $\widehat{f}(A) = \widehat{A}$ then $\widehat{A}(x) = \sup_{x = f^{-1}(y)} A(x)$, for every fuzzy set A over X.

In general the construction of $\widehat{f}$ is not trivial except for f having the property of monotonicity, increasing or decreasing. Defining T as a triangular fuzzy set, denoted by $T = (a,b,c)$, by

$$T(x) = \begin{cases} (x-a)/(b-a) & \text{if } a \leq x \leq b, \\ (c-x)/(c-b) & \text{if } b \leq x \leq c, \end{cases} \tag{1}$$

and $T(x) = 0$ otherwise, let us consider $T = (0, 0, 0.9)$ and $f(x) = x^2$. Hence, the Zadeh extension of f applied to T is the fuzzy set $\widehat{T}$ shown in Figure 1 in the same graphic as T and f.

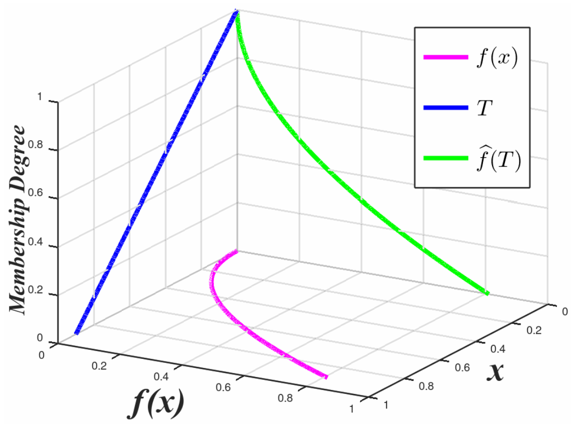

Figure 1: Zadeh extension of the triangular fuzzy number A through the functions $f(x) = x^2$. Graphic by the authors with the aid of Octave [7].

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Usual arithmetic operations can be established using Zadeh's extension principle. The sum of two fuzzy numbers A and B can be obtained in terms of α-levels as

$$[A+B]_\alpha = [L_A(\alpha)+L_B(\alpha), U_A(\alpha)+U_B(\alpha)], \text{ for all } \alpha \in [0,1].$$

The external operation given by the scalar multiplication is given levelwise as

$$[\lambda A]_\alpha = [\lambda L_A(\alpha), \lambda U_A(\alpha)], \text{ if } \lambda \geq 0 \text{ and } [\lambda A]_\alpha = [\lambda U_A(\alpha), \lambda L_A(\alpha)], \text{ if } \lambda < 0, \text{ for all } \alpha \in [0,1].$$

Besides the commutative, associative, and existence of neutral element, other properties of the sum are the quasidistributive, i.e. for all $A, B \in \mathbb{R}_\mathcal{F}$ and $p, q \in \mathbb{R}$, then $(p+q)A = pA + qA$ whenever $pq \geq 0$; if $A \in \mathbb{R}_\mathcal{F} \backslash \mathbb{R}$, then there is no $B \in \mathbb{R}_\mathcal{F}$ such that $A + B = 0$; $(pq)A = p(qA)$ for all $A, B \in \mathbb{R}_\mathcal{F}$ and $p, q \in \mathbb{R}$ [1].

The set $A \in \mathbb{R}_\mathcal{F}$ is symmetric with respect to $x \in \mathbb{R}$ if $A(x-y) = A(x+y)$ for all $y \in \mathbb{R}$, concept that is denoted by $(A \,|\, x)$. The fuzzy number A is nonsymmetric if there is no $x \in \mathbb{R}$ such that this property is satisfied [3, 5].

For a given set $\mathcal{A} = \{A_1, \ldots, A_n\} \subset \mathbb{R}_\mathcal{F}$, the set of all finite linear combinations of $\mathcal{A}$ is denoted by $\mathcal{S}_n(\mathcal{A}) = \left\{ \sum_{n=1}^{n} \xi_i A_i, \xi_i \in \mathbb{R}, i = 1, 2, \ldots, n \right\}$, where the operations involved are the defined by Zadeh's extension principle in $\mathbb{R}_\mathcal{F}$ [3]. A finite set of fuzzy numbers $\mathcal{A} = \{A_1, \ldots, A_n\}$ is called strongly linearly independent (SLI for short) if for all $B \in \mathcal{S}_n(\mathcal{A})$ holds $(B|0)$ implies $\xi_i = 0$ for all $i = 1, \ldots, n$ [3]. The fuzzy set B is called $\mathcal{S}(\mathcal{A})-$ linearly correlated fuzzy number. Moreover, the set $\mathcal{A} \subset \mathbb{R}_\mathcal{F}$ is SLI if, and only if, the function $\psi_n : \mathbb{R}^n \to \mathcal{S}_n(\mathcal{A})$ given by $\psi_n(\xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} \xi_i A_i$ is a bijection [4]. Figure 2 shows an example of a SLI set.

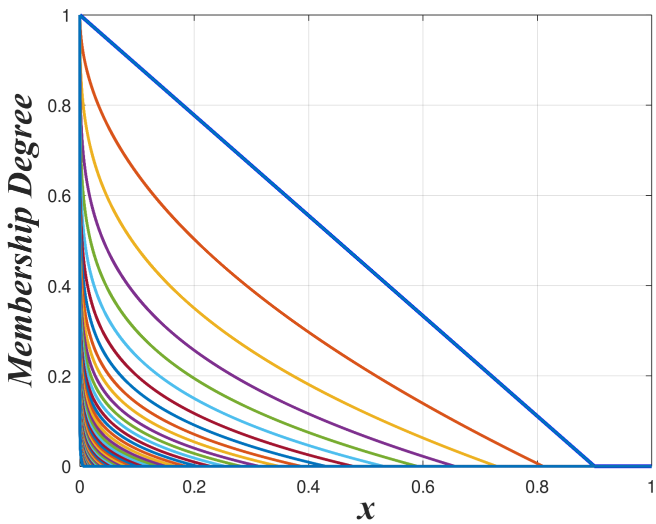


Figure 2: Example of a set of SLI fuzzy numbers composed by the Zadeh extension of the fuzzy number $A = (0, 0, 0.9)$ (blue membership function) through the functions $f(x) = x^n$, where $n \in [0, 50]$. Graphic by the authors with the aid of Octave [7].

From the bijection ψ_n, operations of sum and scalar multiplication are induced in $\mathcal{S}_n(\mathcal{A})$.

Indeed, if $A, B \in \mathcal{A}$ with $A = \sum_{i=1}^{n} \xi_i A_i$ and $B = \sum_{i=1}^{n} \eta_i A_i$, and $\lambda \in \mathbb{R}$ then it is defined:

$$\begin{aligned} A +_{\psi_n} B &= \psi_n\left({\psi_n}^{-1}(A) + {\psi_n}^{-1}(B)\right) = \sum_{i=1}^{n} (\xi_i + \eta_i) A_i \\ -\lambda \cdot_{\psi_n} A &= \psi_n\left(\lambda {\psi_n}^{-1}(A)\right) = \sum_{i=1}^{n} (\lambda \xi_i) A_i. \end{aligned}$$

The induced difference between two $\mathcal{S}(\mathcal{A})-$linearly correlated fuzzy numbers is defined as $A -_{\psi_n} B = A +_{\psi_n} (-1) \cdot_{\psi} B$. It follows that $(\mathcal{S}_n(\mathcal{A}), +_{\psi_n}, \cdot_{\psi_n})$ is a vector space of fuzzy numbers of dimension n. Furthermore, if it is defined $\|A\|_{\psi_n} = \|{\psi_n}^{-1}(A)\|$, $A \in \mathcal{S}_n(\mathcal{A})$ where $\|\cdot\|$ is any norm defined in $\mathbb{R}^n$. It follows that $(\mathcal{S}_n(\mathcal{A}), +_{\psi_n}, \cdot_{\psi_n}, \|\cdot\|_{\psi_n})$ is a finite Banach space, isometric to the Euclidean space of dimension n [4].

The concept of SLI sets has been extended to infinite sets of fuzzy numbers by Laiate and Bertone [11], using the following definition:

$$\begin{array}{l} \mathcal{A} \subset \mathbb{R}_{\mathcal{F}} \text{ is Strongly Linearly Independent} \\ \text{if every finite nonempty subset of } \mathcal{A} \text{ is SLI,} \end{array} \tag{2}$$

Any set $\mathcal{A}$ that verifies the statement of (2) is called an SLI set. For instance, the set shown in Figure 2 is an example of an enumerable SLI extending the construction to infinity.

3 A Hilbert Space of Fuzzy Numbers

Let $\ell^2(\mathbb{R})$ be the Hilbert space of the square summable real sequences, denoted by ℓ^2 and equipped with the norm $\|a\|_{\ell^2} = \left(\sum_{i=1}^{\infty} |a_i|^2\right)^{\frac{1}{2}}$ for any $a = (a_i)_{i\in\mathbb{N}} \in \ell^2$ [9]. One important property of the spaces ℓ^2 is the existence of a Schauder basis, by considering the enumerable set of sequences $\{(e_i^n)_{i\in\mathbb{N}}\}_{n\in\mathbb{N}}$ defined by

$$e_i^n = 1 \text{ if } i = n, \text{ and } e_j^n = 0 \text{ for all } j \neq n. \tag{3}$$

Let $\mathcal{A} = \{A_i\}_{i\in\mathbb{N}}$ be an enumerable SLI set of fuzzy numbers such that $\gamma = (\gamma_i)_{i\in\mathbb{N}} = (\|A_i\|_{\mathcal{F}})_{i\in\mathbb{N}}$ is a sequence belonging to ℓ^2. An example of this set is shown in the example in Figure 2. Indeed, it is noticed that $\|A_i\|_{\psi} = 0.9^i$, thus

$$\sum_{n=1}^{\infty} \gamma^2 = \sum_{n=1}^{\infty} \|A_i\|_{\psi}^2 = \frac{0.9^2}{1 - 0.9^2} = 4.2631,$$

which proves the assertion.

Consider the subset $\mathcal{S}(\mathcal{A})$ of $\mathbb{R}_{\mathcal{F}}$ defined by:

$$\mathcal{S}(\mathcal{A}) = \left\{ B \in \mathbb{R}_{\mathcal{F}} : \exists\ \xi \in \ell^2 \text{ such that } B = \sum_{i=1}^{\infty} \xi_i A_i \right\} \tag{4}$$

where the convergence considered is given w.r.t. $\mathcal{D}_\infty$.

Notice that the set $\mathcal{S}(\mathcal{A})$ as defined in (4) is not empty. In fact, taking the sequence of the example illustrated in Fig. 2, the enumerable family $\{e^n\}_{n\in\mathbb{N}}$, Schauder basis of ℓ^2 [11], and defining $A_n = \sum_{i=1}^{\infty} e_i^n A_i$, it is found an enumerable family of elements of $\mathcal{S}(\mathcal{A})$.

Extending the bijection ψ_n for infinite dimensional spaces it is shown that considering $\mathcal{S}(\mathcal{A})$ defined in (4), and the function $\psi_\gamma : \ell^2 \to \mathcal{S}(\mathcal{A})$ such that, if $B = \sum_{i=1}^{\infty} \xi_i A_i$, for some $\xi = (\xi_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{R})$ then $\psi\gamma(\xi) = B$, thus ψ_γ is a bijection if and only if the set $\mathcal{A}$ is SLI. Proceeding in the same manner as in the finite case, induced operations denoted by $+_{\psi_\gamma}$ and $\cdot_{\psi_\gamma}$ are defined, and a norm denoted by $\|\cdot\|_{\psi_\gamma}$ is induced from an inner product, denoted by $\langle\cdot,\cdot\rangle_{\psi_\gamma}$, that provides the set $\mathcal{S}(\mathcal{A})$ the structure of a Hilbert space [11].

4 Fourier Series of Fuzzy Numbers

Based on the theoretical background presented in the previous section, a fuzzy Fourier series theory and its corresponding real counterpart can be developed, which is necessary for real-world problems where uncertainties are present. Indeed, given a piecewise smooth periodic real function f with period $2L$. Its Fourier series on $[-L, L]$ is given by:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

which coefficients belong to ℓ^2 and have the form of

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx,\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, n \geq 1.$$

Then, the Fourier series converges to $\dfrac{f(x^+) + f(x^-)}{2}$, where $f(x^\pm) = \lim_{\epsilon\to 0} f(x \pm \epsilon)$.

Using the bijection $\psi_n : \mathbb{R}^n \to \mathcal{S}_n(\mathcal{A})$, where $\mathcal{A} = \{1, A_1, \ldots, A_{n-1}\}$ is SLI and $A_i \in \mathbb{R}_\mathcal{F}\backslash\mathbb{R}$ for all $i = 1, \ldots, n-1$, it is possible to express the corresponding approximations of f in terms of fuzzy numbers. In fact, let $a_i,\ b_i \in \ell^2$ and the corresponding images through ψ_n, namely A_n and B_n a fuzzy Fourier approximation of f of order $n-1$ is image through ψ_n of the trigonometric polynomial of order $n-1$ as follows:

$$\begin{aligned} &\psi_n\left(\frac{a_0}{2}, a_1\cos\left(\frac{\pi t}{L}\right) + b_1\sin\left(\frac{\pi t}{L}\right), \ldots, a_{n-1}\cos\left(\frac{(n-1)\pi t}{L}\right) + b_{n-1}\sin\left(\frac{(n-1)\pi t}{L}\right)\right) \\ &= \frac{a_0}{2} + \sum_{i=1}^{n-1}\left(a_i\cos\left(\frac{i\pi t}{L}\right) + b_i\sin\left(\frac{i\pi t}{L}\right)\right)A_i. \end{aligned} \tag{5}$$

It is well known that this theory applies in signal processing, specifically in approximating signals using the Fourier series [12]. Indeed, suppose a periodic signal, e.g., a square wave or an audio signal with finite energy. The coefficients of these waves form a sequence in ℓ^2, ensuring that the approximation converges to the original signal. In particular, let us consider the square wave with period $\frac{\pi}{4}$ given by:

$$g(t) = \begin{cases} 1 & \text{if } 0 \leq t < \frac{\pi}{4}, \\ 0 & \text{if } \frac{\pi}{4} \leq t < \frac{\pi}{2}. \end{cases}$$

Expanding into odd harmonic Fourier series, the coefficients $b = (b_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{R})$ as $b_{2i-1} = \frac{2}{(2i-1)\pi}$. Moreover, note that the following equalities hold:

$$\sum_{i=1}^{\infty} |b_i|^2 = \left(\frac{4}{\pi}\right)^2\left(1 + \sum_{i=1}^{\infty}\frac{1}{(2i+1)^2}\right) \quad \text{and} \quad \sum_{i=1}^{\infty}\frac{1}{(2i+1)^2} = \frac{\pi^2}{8}.$$

Therefore,

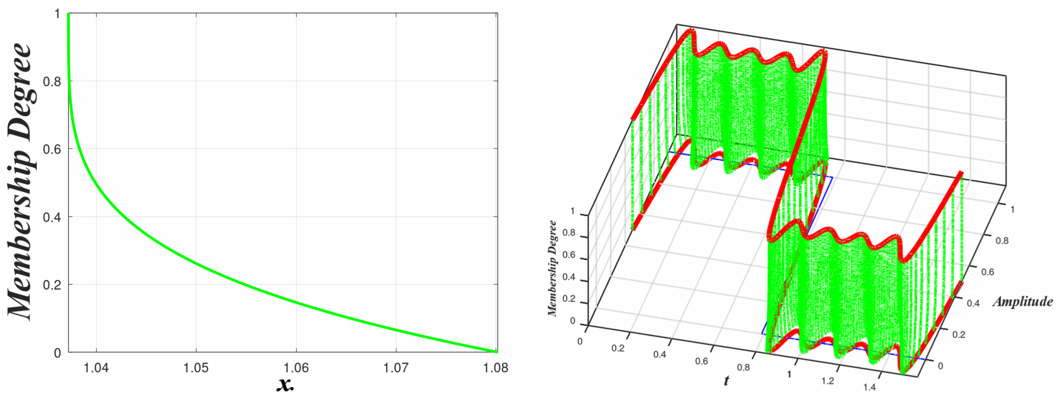
$$\sum_{i=1}^{\infty}|b_i|^2 = \left(\frac{16}{\pi^2}\right)\left(\frac{\pi^2}{8}\right) = 2 < \infty,$$

which confirms $b \in \ell^2(\mathbb{R})$. Using the approximation error in the norm $g(t)$ with the first terms N using the formula $g_N(t) = \sum_{i=1}^{N} b_i \sin((2i+1)t)$, the approximation error in the norm ℓ^2 is given by $\|g - g_N\|^2 = \sum_{i=N+1}^{\infty} |b_i|^2$. Since $b \in \ell^2(\mathbb{R})$, the error $\|g - g_N\|_{\ell^2} \to 0$ is as $N \to \infty$. The approximation for $N = 4$ is shown in Figure 4 (b) on the plane $z = 1$.

The fourth Fourier approximation is constructed in the space $\mathcal{S}_5(\mathcal{A})$ corresponding to the enumerable SLI family, the first element being the singleton 1. The approximation is made through the bijection ψ_n and the fourth first Fourier terms:

$$\psi_5\left(0.5, b_1\sin(3t), b_2\sin(5t), b_3\sin(7t), b_4\sin(9t)\right) = 0.5 + \sum_{i=1}^{4} b_i \sin((2i+1)t)A_i,$$

for each $t \in [0, \pi/2]$. The simulation result is depicted at a fixed time in Figure 4 (a) and for all time $t \in [0, \frac{\pi}{2}]$ in Figure 4 (b).



(a) (b)

Figure 3: (a) Fuzzy Fourier approximation at fixed time. (b) Fourth Fourier approximation of wave g in the space $\mathcal{S}(\mathcal{A})$. The projection on the plane z = 1 represents the real Fourier wave, which forms the core of the fuzzy approximation. Graphic by the authors with the aid of Octave [7].

5 Final Remarks

This manuscript discusses Banach and Hilbert spaces of infinite dimension of fuzzy numbers. In particular, via a bijection $\psi_\gamma : \ell^2 \to \mathcal{S}(\mathcal{A})$, an identification of a subspace of real-valued sequences with convergent square sum with a subset of fuzzy numbers, the structure of Hilbert space is inherited by $\mathcal{S}(\mathcal{A}) \subset \mathbb{R}_{\mathcal{F}}$. In addition to induced arithmetic operations, an induced inner product defines the infinite-dimensional fuzzy Hilbert space.

As an introductory depiction of the theory, it presents an example of pointwise approximation of a piecewise continuous real-valued function, namely, the square wave function. The approximation using $\mathcal{S}(\mathcal{A})-$linearly correlated fuzzy functions of a classic function is measured by the decreasing fuzziness of the fuzzy Fourier series. In light of classical theory, this discussion will further explore uniform approximations of piecewise continuous fuzzy functions, where continuity is defined with respect to both Hausdorff and induced norms.

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