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Estimation of the Saving Rate Function in the Spatial Solow-Swan Model by PINNs

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The Solow-Swan model, developed in 1956 [6, 7], has been a key reference for understanding economic growth, but it does not account for the spatial distribution of activities, limiting its applicability. To address this limitation, spatial dimensions have been incorporated [1, 4], allowing researchers to study the impact of location on growth. Recently, researchers have been exploring inverse problems, such as obtaining the convex-concave production function in the spatial model [2, 3], but to the best of our knowledge, no study has yet investigated the determination of the saving rate function.

In this context, this work aims to determine the saving rate, which depends solely on the spatial coordinate, by solving the inverse problem in the spatial Solow-Swan model. We propose a model that describes the evolution of capital density in local economies distributed along the compact interval $\Omega = [0, l]$, with $0 < l < \infty$. At each point $x \in \Omega$, there is a capital density $K(t, x) \geq 0$ and a labor density $L(x) \geq 0$, which are used to produce an aggregate good through a Cobb-Douglas production function

$$f(K, L) = A(x)[K^{\phi}(t, x)L(x)^{1-\phi}]. \tag{1}$$

Here, $A(x)$ represents the technological factor, and $\phi \in (0, 1)$ indicates the intensity of capital usage. The distribution of labor is initially given by the exogenous function $L(x) \geq 0$. Thus, the evolution of the capital stock in the economy is governed by the following reactive-diffusive partial differential equation, with the corresponding initial and boundary conditions

$$K_t = s(x)A(x)\left[K^{\phi}L(x)^{1-\phi}\right] - \delta(x)K + d(x)K_{xx},\ x \in (0, l),\ t > 0, \tag{2a}$$

$$K(t, x) = K_0(x),\ x \in \Omega = [0, l],\ t = 0, \tag{2b}$$

$$K_x = 0,\ x \in \partial\Omega = \{0, l\},\ t > 0. \tag{2c}$$

Here, $s(x) \in (0, 1)$ is the saving rate function that we aim to estimate, $\delta(x) \in (0, 1)$ is the capital depreciation rate, and $d(x) > 0$ is the capital diffusion coefficient, which indicates the intensity of capital movement to regions with less available capital. Completing the model, the initial capital distribution, $K_0(x) \geq 0$, is given by (2b), while the homogeneous Neumann boundary conditions (2c) ensure no transfer of capital and labor at $\partial\Omega$, making the economy a closed economy.

We propose a methodology based on *Physics-Informed Neural Networks* [5] with two *Multi-Layer Perceptron* networks. The first network solves the PDE in (2), using t and x as inputs, while the second estimates the saving rate with x as the only input. Both networks use the Adam optimizer with a learning rate of 10^{-3} to calibrate the parameters (weights and biases) of the networks. Studies were conducted to determine the architecture of each network. We concluded with a network that solves the PDE with a 2-layer deep architecture and 70 neurons, while the

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one approximating the saving rate function uses a 1-layer deep architecture and 40 neurons. In both networks, the activation functions up to the last layer are hyperbolic tangent. In the last layer, the first network uses the *softplus* activation function to ensure non-negative values, which aligns with the nature of capital, while the second network uses the hyperbolic tangent. Finally, we identified the ideal input sample set for x and t in the first network, consisting of 2500 points, which achieves five consecutive iterations with the loss function below 10^{-5} in the fewest epochs.

To evaluate the methodology, we consider a case study with (2): $l \equiv 10$, $T \equiv 10$, $\phi \equiv 1/3$, $\delta(x) \equiv 0.05$, $A(x) \equiv 1$, $d(x) \equiv 0.25$, and the functions $L(x) = 0.3x^2\left[1 - \cos\left(4\pi x/l\right)\right]$, $k_0(x) = \cos^4\left(\pi x/(2l) - \pi/2\right)$, and $s(x) = 0.2\cos^4\left(\pi x/(2l)\right)$. Data for the inverse problem consisted of discrete K(x,t) measurements produced by a finite-element solver of the manufactured direct problem. We have obtained estimations of $s(x)$ with a mean squared error of $8 \cdot 10^{-6}$. Using PINNs in the direct approach and comparing them with the finite element method, the error was 10^{-5}. For the inverse problem, with 10 validation points, we obtained errors of $2 \cdot 10^{-5}$ for $k(x,t)$ and $5 \cdot 10^{-6}$ for $s(x)$, which are considered satisfactory results. Further work should extend test cases and apply real data from IBGE.

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