Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 12, n. 1, 2026.

Trabalho apresentado no XLIV CNMAC, Fundação Getulio Vargas - Rio de Janeiro - RJ, 2025.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Punishing Costs in Compulsory Public Goods Games

Rocío Botta[1]
Polytechnical School - National University of Asunción, San Lorenzo, Paraguay
Gerardo Blanco[2]
Pontificia Universidad Católica de Valparaíso, Valparaiso, Chile
Christian E. Schaerer[3]
Polytechnical School - National University of Asunción, San Lorenzo, Paraguay

In public goods games, the principal obstacle to obtaining cooperation is the temptation to be a free-rider, this is, to benefit from the game without contributing [7]. In order to reduce defection, incentives are applied [1]; however, rewards or punishment are costly, and therefore, difficult to implement [4, 5]. To address this problem, one option is to sanction only a subset d of the defectors in the population; this mechanism, called fractional punishment [3], reduces the cost while improving cooperation.

Nonetheless, obtaining the resources to implement the punishment remains a challenge. In this work, in a compulsory public goods game with fractional punishment, a percentage of the contribution to the public good is set apart for punishing expenses and the effect on the cooperation of the group is analyzed. Following [2, 6], the payoffs in the game for cooperators and defectors are as follows:

$$p_y = (1-d)\frac{rc}{n}(n-1)x + d(1-u)\frac{rc}{n}(n-1)x, \tag{1}$$

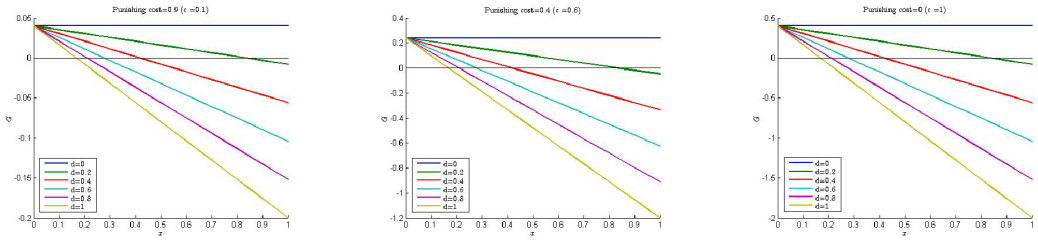
$$p_x = \frac{rc}{n}(n-1)x + \frac{rc}{n} - c, \tag{2}$$

where c is the contribution to the game, r is the multiplication factor, n is the number of players, $0 \leq d \leq 1$ is the subset of defectors punished and $0 \leq u$ is the amount discounted from their payoff. The d subset of the defectors will have their payoff reduced by u while the $1-d$ subset will obtain their normal payoff. The frequency of cooperators is represented by x and $1-x$ is the frequency of defectors; since $x + y = 1$, the system can be represented with one equation:

$$\dot{x} = -x(1-x)(p_y - p_x). \tag{3}$$

In the simulation made using Octave, the payoffs for cooperators and defectors were calculated for $0 \leq x \leq 1$. The payoff difference between both strategies is represented by the function $G = p_y - p_x$. When $G < 0$ the outcome is full cooperation; if $G = 0$ both strategies have the same payoff, and if $0 < G$ the outcome is full defection. As can be noticed in Figure 1, as punishing expenses increases, in other words, if the percentage of the contribution set aside for punishment increases, the payoffs for both strategies decreases; however, higher values of d produce higher payoffs and full cooperation can be achieved with a lower percentage of cooperation (x) in the population.

[1] rbotta@pol.una.py
[2] gerardo.blanco@pucv.cl
[3] cshaerer@pol.una.py



(a) $c = 0.1$ (b) $c = 0.6$ (c) $c = 1$

Figure 1: Punishing cost with fractional punishment for several values of d and $0 \leq x \leq 1$. Parameters: $u = 1$, $r = 3$ and $n = 5$. (a) $c = 0.1$, (b) $c = 0.6$, and (c) $c = 1$. Source: Elaborated by the authors.

Aknowledgment

Authors thank the project PINV01-694 “Simulación y control en juego de bienes públicos: El problema del free-rider y el castigo fraccionado" CONACYT-FPUNA.

References

[1] J. Andreoni, W. Harbaugh, and L. Vesterlund. “The Carrot or the Stick: Rewards, Punishments, and Cooperation”. In: **American Economic Review** 93.3 (June 2003), pp. 893–902. DOI: 10.1257/000282803322157142.

[2] R. Botta, G. Blanco, and C. E. Schaerer. “Discipline and punishment in panoptical public goods games”. In: **Scientific Reports** 14 (2024), p. 7903. DOI: 10.1038/s41598-024-57842-0.

[3] R. Botta, G. Blanco, and C. E. Schaerer. “Fractional Punishment of Free Riders to Improve Cooperation in Optional Public Good Games”. In: **Games** 12.1 (2021). DOI: 10.3390/g12010017.

[4] X. Chen, A. Szolnoki, and M. Perc. “Probabilistic sharing solves the problem of costly punishment”. In: **New Journal of Physics** 16 (2014). DOI: 10.1088/1367-2630/16/8/083016. arXiv: 1408.1945.

[5] F. Dercole, M. De Carli, F. Della Rossa, and A. V. Papadopoulos. “Overpunishing is not necessary to fix cooperation in voluntary public goods games.” In: **Journal of theoretical biology** 326 (2013), pp. 70–81. DOI: 10.1016/j.jtbi.2012.11.034.

[6] C. Hauert, S. De Monte, J. Hofbauer, and K. Sigmund. “Replicator Dynamics for Optional Public Good Games”. In: **Journal of Theoretical Biology** 218.2 (2002), pp. 187–194. DOI: 10.1006/jtbi.2002.3067.

[7] P. Kollock. “Social Dilemmas: The Anatomy of Cooperation”. In: **Annual Review of Sociology** 24 (1998), pp. 183–214. ISSN: 03600572, 15452115.