Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 12, n. 1, 2026.

Trabalho apresentado no XLIV CNMAC, Fundação Getulio Vargas - Rio de Janeiro - RJ, 2025.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

On the Use of Explicit Eigenfunctions in the Calculation of Multiplicity Moments in Nuclear Safeguards

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Recent work [3] has shown the application of the particle transport equation to calculate the so-called multiplicity moments, which are very important in nuclear safeguards. In [3], a transformation was made from the integral formulation of the problem in a sphere to a slab geometry problem, where the Analytical Discrete Ordinates (ADO) method, a spectral method, was applied [1]. Numerical eigenfunctions were used, defined in terms of eigenvectors.

In this work, we investigate the use of explicit expressions for the eigenfunctions that define the ADO solution. We are interested in precise formulations that require low computational time, as these results shall be used to solve inverse problems.

The one-dimensional neutron transport equation, in plane geometry, with azimuthal symmetry and isotropic scattering is given by [1]

$$\mu \frac{\partial}{\partial \tau}\psi(\tau,\mu) + \psi(\tau,\mu) = \frac{c}{2}\int_{-1}^{1}\psi(\tau,\mu')d\mu' + S(\tau), \tag{1}$$

where $\tau \in (0,\tau_0)$ is the optical variable and $\mu \in [-1,1]$ is the cosine of the direction of propagation, $c \in (0,1)$ is the average number of neutrons per collision and $\psi(\tau,\mu)$ is the angular flux of neutrons.

By discretizing the range of the angular variable into N nodes $\mu_i \in (0,1)$ and N weights w_i, it is possible to express the solution in discrete ordinates as [1]

$$\psi(\tau,\pm\mu_i) = \sum_{j=1}^{N} A_j\phi(\nu_j,\pm\mu_i)e^{-\tau/\nu_j} + B_j\phi(\nu_j,\mp\mu_i)e^{-(\tau_0-\tau)/\nu_j} + \psi_p(\tau,\pm\mu_i), \tag{2}$$

where ν_j are the eigenvalues, $\phi(\nu_j,\pm\mu_i)$ are the associated eigenfunctions, A_j and B_j are the superposition coefficients of the homogeneous solution, and $\psi_p(\tau,\pm\mu_i)$ refers to the particular solution derived via Green's Function [2].

The explicit eigenfunctions can be computed using the following expression [1]

$$\phi(\nu_j,\mu_i) = \frac{c\nu_j}{2}\frac{1}{\nu_j-\mu_i}, \tag{3}$$

while the numerical eigenfunctions are written from eigenvectors as [1]

$$\Phi_{\pm}(\nu_j) = \frac{1}{2}M^{-1}(I_N \pm \nu_j A)X(\lambda_j), \tag{4}$$

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where I_N is the identity matrix of order N, M^{-1} is a diagonal matrix whose entries are the reciprocal of the nodes, $X(\lambda_j)$ is the eigenvector and $\Phi_{\pm}(\nu_j)$ is the vector whose components are eigenfunctions associated to the eigenvalues ν_j.

In Table 1, we present results for the uniformly distributed source induced moments, N^* (*Singles*), M^* (*Doubles*), W^* (*Triples*) [4] which depend on the calculation of the single particle induced moments: the first moment $n(\tau)$, the second moment $m(\tau)$ and the third moment $w(\tau)$[3]. These, in turn, are obtained by solving equations such as (1). For example, the first moment $n(\tau)$ is given in terms of the ADO solution as [3]

$$n(\tau) = \frac{1}{\tau}\sum_{j=1}^{N}[A_j e^{-\tau/\nu_j} + B_j e^{-(\tau_0-\tau)/\nu_j}]\sum_{k=1}^{N} w_k[\phi(\nu_j,\mu_k) + \phi(\nu_j,-\mu_k)]. \tag{5}$$

The quantities N^*, M^* and W^* are given by [4]

$$N^* = \frac{3\nu_{s,1}}{\tau_0^3}\int_0^{\tau_0}\tau^2 n(\tau)d\tau, \tag{6}$$

$$M^* = \frac{3}{\tau_0^3}\int_0^{\tau_0}\tau^2\{\nu_{s,2}n^2(\tau) + \nu_{s,1}m(\tau)\}d\tau, \tag{7}$$

$$W^* = \frac{3}{\tau_0^3}\int_0^{\tau_0}\tau^2\{\nu_{s,3}n^3(\tau) + 3\nu_{s,2}n(\tau)m(\tau) + \nu_{s,1}w(\tau)\}d\tau, \tag{8}$$

In this work, we compare the two approaches to eigenfunctions. In both cases, we obtained the same numerical results, although with the use of explicit eigenfunctions there was an improvement in the computational time of at least 10%. We used an angular discretization of $N = 20$ nodes using the Gauss-Legendre Quadrature. For the evaluation of the particular solution and the calculation of the factorial moments, equation (6) to (8), we used the Gauss-Legendre Quadrature with 40 nodes. From the calculated quantities N^*, M^* and W^*, we want to estimate the fissile material in a spherical sample.

Table 1: Moments N^*, M^* and W^*, $\tau_0 = 1$, c = 0.9.

N^*	M^*	W^*
1.4231192	4.0841080	29.468598

Acknowledgements

The authors would like to thank CNPq of Brazil for partially funding this work.

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