Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 12, n. 1, 2026.

Trabalho apresentado no XLIV CNMAC, Fundação Getulio Vargas - Rio de Janeiro - RJ, 2025.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Importance Sampling for Logarithmic Pool Ensembles

Atílio L. Pellegrino[1], Luiz M. Carvalho[2]
FGV/EMAp, Rio de Janeiro, RJ

In this work, theoretical results and computational simulations are being developed for the use of Importance Sampling (IS) schemes in situations where the target distribution ($\overline{\pi}_\alpha(x)$) can be expressed as proportional to a logarithmic combination of distributions ($\pi_k(x)$'s):

$$\overline{\pi}_\alpha(x) = t(\alpha)\prod_{k=1}^{K}\pi_k(x)^{\alpha_k} = \begin{cases} t(\alpha)\exp\left(\sum_{k=1}^{K}\alpha_k\log\left(\pi_k(x)\right)\right), & \text{if } \forall\, k \quad \pi_k(x) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The objectives include determining the optimal proposals sets and bounds for the variances of the IS estimators when calculated using these proposal sets and comparing them to known proposal schemes.

In addition to evaluating, via computational experiments, whether the loss of efficiency when sampling from the set of proposals (compared to that obtained using $q_{opt}(x)$) is relevant or not, remembering that it is not always easy to sample from the optimal proposal.

A review of properties and results for logarithmic pooling can be found in [1] and for Importance Sampling estimators can be found in [4], where it is also shown that for a simple case, for example, in which $f(x) = x$ and the target is $\overline{\pi}(x) \sim N(-1, 1)$ we have as the optimal proposal (for the Importance Sampling estimator) $q_{opt}(x) \propto |f(x)|\overline{\pi}(x)$, which will be a bimodal distribution, that is, in many cases sampling from the optimal proposal is not always easy and possible, therefore, one way is to identify a set of normal proposals that minimizes the variance and allows the sampling process to become easier, always remembering that while there is a gain in this efficiency in sampling, something is lost in the efficiency of the estimator (for example, an increase in variance).

In a review on combination of expert's opinions [2] and [5] present the use of mathematical aggregations using linear opinion pool and log opinion pool, which are respectively a linear combination of distributions and a logarithmic combination of distributions. In this context, each distribution $\pi_k(x)$ becomes the distribution attributed by an expert (the $k-th$ expert in a group of K experts) about an event and the pool is the combined distribution of opinions, in this scenario the weights (α_k's) would represent the weight given to the expert's opinion. Log pooling is *externally Bayesian* (i.e., given new information, if update each of the experts' distributions $\pi_k(x)$'s and then combine them obtain the same distribution if, with the new information, update the distribution $\overline{\pi}_\alpha(x)$), however, it does not satisfy *marginalization property* unlike linear pooling, but this does not satisfy *externally Bayesian*.

In [6] and [7] can be found a review of some results and types of Importance Sampling, such as: Adaptive Parametric Importance Sampling, Sequential Importance Sampling, Annealed Importance Sampling, Multiple Importance Sampling (MIS). A unified framework for MIS (when you have samples for more than one proposal and combine then to estimate the target) schemes is show in [3]. The authors present theoretical descriptions of possible combinations of sampling and weighting procedures when there are sets of proposals (distributions) and from these one wants to produce approximations of the target via IS. In addition, the authors present theoretical results

[1]atilio.lpellegrino@gmail.com
[2]luiz.fagundes@fgv.br

about the variance of the analyzed estimators and perform some numerical experiments showing the performance of the sampling and weighting schemes.

If the target is normal, it is possible to obtain normal samples where their logarithmic combination results in the target, it suffices to see that in the case where our set of K distributions are normal ($\pi_k(x) \sim N(\mu_k, \sigma_k^2)$), our target would be $\overline{\pi}_\alpha(x) \sim N(\mu^*, (\sigma^*)^2)$, where:

$$\mu^* = \frac{\sum_{k=1}^{K} \frac{\alpha_k}{\sigma_k^2}\mu_k}{\sum_{k=1}^{K} \frac{\alpha_k}{\sigma_k^2}}, \quad (\sigma^*)^2 = \frac{1}{\sum_{k=1}^{K} \frac{\alpha_k}{\sigma_k^2}}. \tag{2}$$

Thus, we have an ideal situation: in which both the target is a known distribution and, by fixing the weights, we know the set of normal proposals that generates it and, through computational experiments, we can compare the performance of the estimation via sampling of the set of proposals with other proposal schemes already explored in the literature. We also seek to find bounds for the variance of the IS estimator generated from the logarithmic pool of proposals, focusing on the case of normal proposals but, whenever possible, trying to obtain more general results.

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