Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 12, n. 1, 2026.

Trabalho apresentado no XLIV CNMAC, Fundação Getulio Vargas - Rio de Janeiro - RJ, 2025.

Adaptive System Identification for GMRES(m) Based on DMDc

C. Benitez[1] J. Cabral[2] Christian E. Schaerer[3]
National University of Asuncion, San Lorenzo, Central, Paraguay

The Restarted Generalized Minimal Residual Method (GMRES(m)) is a widely used iterative solver for large-scale, sparse, and nonsymmetric linear systems [4]. However, its convergence is not guaranteed [1]. To improve convergence, the restart parameter m must be properly tuned; however, there is currently no general rule for selecting its optimal value. The primary challenge in designing a rule is the internal nonlinear dynamics of GMRES(m), which is well understood in its construction, but less well understood in its behavior. In this work, we design a system identification model (SIM) to address the challenge of capturing the internal nonlinear behavior of GMRES(m). The results show the efficiency of the SIM as an observer of the internal dynamics. Future works will be oriented towards using the SIM to enhance the convergence of the method.

Let $Ax = b$ be the linear system with $n \times n$-matrix A, and n-dimensional vectors x and b. Let m be the restarting parameter input of GMRES(m) corresponding to the Krylov subspace dimension built on the matrix A and the initial residual r_0 given by $\mathcal{K}_m(A, r_0) = span\{r_0, Ar_0, \ldots, A^{m-1}r_0\}$. A cycle of GMRES($m$) can be represented by a discrete dynamical system that, starting from an approximate solution x_j, generates another approximate solution $x_{j+1} \leftarrow GMRES(x_j, m)$. Hence, from an initial approximate solution x_0, we collect p cycles of approximate solutions of GMRES(m) in matrices $\mathbf{X} = [x_0 \ldots x_{p-1}]$ and $\mathbf{X}' = [x_1 \ldots x_p]$ (both with dimension $n \times p$), and the restarting parameters m collected in a matrix $\mathbf{\Upsilon} = [m_0 \ldots m_{p-1}]$. Hence it is obtained

$$\mathbf{X}' \approx H\mathbf{X} + B\mathbf{\Upsilon}. \tag{1}$$

where H and B are matrices to be determined. To this end, the expression (1) is rewritten as $\mathbf{X}' \approx G\,\Omega$ where $G = [H\ B] \in \mathbb{R}^{n\times(n+1)}$ and $\Omega = [X^T\ \Upsilon^T]^T$ with $\Omega \in \mathbb{R}^{(n+1)\times p}$. The best-fit operator G is given as: $G = \arg\min_G \|\mathbf{X}' - G\Omega\|_F = X'\,\Omega^\dagger$ where $\|\cdot\|_F$ is the Frobenius norm and † denotes the pseudo-inverse. Then, since for each measurement trio $\mathbf{X}$, $\mathbf{X}'$, and $\mathbf{\Upsilon}$; Ω contains both the measurement and control snapshot information, then matrices H and input matrix B are determined by structurally corresponding to operator G [3]. Every p cycles the model (1) is recomputed. Hence, two main questions arise: (a) what should be the size p (i.e., the number of cycles in a batch) to be collected in $\mathbf{X}$ and (b) how can p be updated adaptively to maintain a good approximation of the GMRES(m) dynamics. In this work, $p^{(j)}$ denotes the number of cycles in the jth batch and is computed adaptively according to

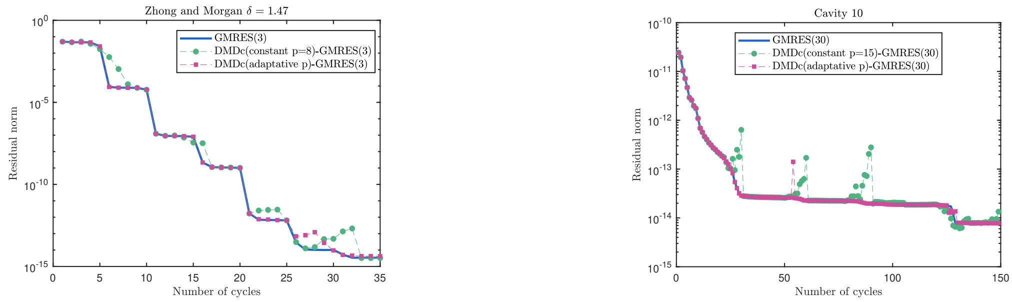
$$p^{(j)} = p_{\min} + \left\lceil (p_{\max} - p_{\min}) \frac{\text{Disp}^{(j-1)}}{\text{Disp}_{\max}} \right\rceil, \quad j = 1, \ldots, N-1. \tag{2}$$

In this formula, the dispersion of the GMRES(m) approximations in the $(j-1)$th batch is measured by $\text{Disp}^{(j-1)} = \sum_{k=1}^{p^{(j-1)}} \left\| x^{(j-1)}_{\text{GMRES},k} - x^{(j-1)}_{\text{GMRES},k-1} \right\|$ where $x^{(j-1)}_{\text{GMRES},k}$ is the kth approximation in

[1] carlosbenitez@fiuna.edu.py
[2] jcabral@fiuna.edu.py
[3] cschaer@pol.una.py

that batch, and $\text{Disp}_{\max}$ is a reference value set as the maximum dispersion observed in previous batches. This adaptive strategy dynamically adjusts the batch size $p^{(j)}$ based on the convergence behavior observed in the preceding batch.

To illustrate the potential of the adaptive DMDc to capture the internal dynamics of GMREs(m), we present two examples chosen by its difficulty of convergence. Figure 1 shows two simulation scenarios employing benchmark matrices from the SuiteSparse Matrix Collection repository [2]. For the Zhong and Morgan [5] benchmark (see subfigure (a)), the parameters are: $m = 3$, tol $= 10^{-15}$, $p_{\min} = 5$, $p_{\max} = 15$, and $x_0 = 0$. For the Cavity matrix [2] (see subfigure (b)), the parameters are set as follows: $m = 30$, tol $= 10^{-15}$, $p_{\min} = 4$, $p_{\max} = 20$, and $x_0 = 0$. In both cases, the GMRES(m) (in blue) is compared with the DMDc norms residuals, showing results with constant p (in green) and adaptively chosen p (in magenta) using equation (3).



(a) Zhong and Morgan

(b) Cavity 10

Figure 1: Comparison of GMRES(m) and DMDc residual norms for benchmark matrices.

Observe that, depending on the choice of the parameter p in DMDc, a simpler adaptive linear dynamical model (1) captures the nonlinear dynamics of GMRES(m) when p is chosen adaptively. This is good news, as new control laws can be designed based on the linear model to enhance the convergence of GMRE(m) by using the system identification model DMDc. This is a topic for the next work.

References

[1] J. C. Cabral, C. E. Schaerer, and A. Bhaya. "Improving GMRES(m) using an adaptive switching controller". In: **Numerical Linear Algebra with Applications** 27.5 (2020), e2305. DOI: 10.1002/nla.2305.

[2] S. P. Kolodziej, M. Aznaveh, M. Bullock, J. David, T. A Davis, M. Henderson, Y. Hu, and R. Sandstrom. "The suitesparse matrix collection website interface". In: **Journal of Open Source Software** 4.35 (2019), p. 1244. DOI: 10.21105/joss.01244.

[3] J. L. Proctor, S. L. Brunton, and J. Nathan Kutz. "Dynamic mode decomposition with control". In: **SIAM Journal on Applied Dynamical Systems** 15.1 (2016), pp. 142–161. DOI: 10.1137/15M1013857.

[4] Y. Saad. **Iterative Methods for Sparse Linear Systems**. 2nd ed. Philadelphia: Society for Industrial and Applied Mathematics, 2003. ISBN: 9780898715347. URL: https://epubs.siam.org/doi/book/10.1137/1.9780898718003.

[5] B. Zhong and R. Morgan. "Complementary cycles of restarted GMRES". In: **Numerical Linear Algebra with Applications** 15.6 (2008), pp. 559–571. DOI: 10.1002/nla.589.

