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ANN-Flux Method for Nonconvex Fluxes in Riemann Problems

Marcelo M. Mello[1] Arthur M. Espírito Santo[2] Pedro H. A. Konzen[3]
PPGMAp/IME/UFRGS, Porto Alegre, RS

Artificial neural networks (ANNs) [3] have been successfully applied to solve partial differential equations, particularly since the emergence of physics-informed neural networks (PINNs) [7]. Applications to nonlinear conservation laws have also shown promising results, including PINNs for high-speed flows [6], conservative PINNs [4], and weak PINNs [8]. In this work, we propose a novel method, called ANN-Flux, to solve the Riemann problem for the nonlinear scalar conservation law

$$u_t + (F(u))_x = 0, \quad x, t \in \mathbb{R} \times (0, t_f], \tag{1}$$

$$u(x,0) = u_L, \quad x \in \mathbb{R}_-, \quad u(x,0) = u_R, \quad x \in \mathbb{R}_+, \tag{2}$$

with prescribed left and right states $u_L, u_R \in \mathbb{R}$, and a flux function $F : \mathbb{R} \to \mathbb{R}$. When the flux function is nonconvex, the entropy solution of problem (1) may consist of a combination of shock and rarefaction waves, depending on the initial states. The two canonical cases are:

$$u(x,t) = \begin{cases} u_L, & x < t\sigma, \\ u_R, & x > t\sigma, \end{cases} \text{ (shock)} \quad u(x,t) = \begin{cases} u_L, & x < tF'(u_L), \\ G(x/t), & tF'(u_L) < x < tF'(u_R), \\ u_R, & x > tF'(u_R), \end{cases} \text{ (raref.)} \tag{3}$$

where $G = [F']^{-1}$ and the shock speed satisfying the Rankine–Hugoniot condition is $\sigma = \frac{F(u_L)-F(u_R)}{u_L-u_R}$. The ANN-Flux (ANN-F) method is designed for intricate flux functions that are computationally expensive to evaluate, differentiate, or invert. Given that neural networks are universal approximators [3], ANN-F approximates the flux F using a multilayer perceptron (MLP) $\tilde{y} = \mathcal{N}_F(u)$ trained to minimize the mean squared error (MSE) loss $\varepsilon\left(\tilde{y}^{(s)}, y^{(s)}\right)$ on the dataset $\{\left(u^{(s)}, y^{(s)} = F(u^{(s)})\right)\}_{s=1}^{n_{s,F}}$, where $n_{s,F}$ denotes the number of training samples. For the shock wave case, the solution is obtained by replacing F with $\mathcal{N}_F$ in the left-hand expression of (3). In the rarefaction case, however, a second set of MLPs, denoted by $\mathcal{N}_G^{(k)}$, is introduced to approximate the inverse of the derivative of F, i.e., $[\mathcal{N}_F']^{-1} \approx [F']^{-1}$. The derivative $\mathcal{N}_F'$ is efficiently computed via automatic differentiation [2]. Each network $\mathcal{N}_G^{(k)}$ is trained using a dataset of the form $\{(\delta\tilde{y}^{(s)}, u^{(s)})\}_{s=1}^{n_{s,G}}$, where $\delta\tilde{y}^{(s)} = \mathcal{N}_F'(u^{(s)})$ and $u^{(s)}$ is sampled in subdomains of $[u_L, u_R]$ defined by changes in concavity. Specifically, the domain is partitioned as $\{u_L \le u^{(s)} \le u_m^1\}$, $\{u_m^1 \le u^{(s)} \le u_m^2\}$, $\ldots$, $\{u_m^k \le u^{(s)} \le u_R\}$, where $\{u_m^1, u_m^2, \ldots, u_m^k\}$ are the points of inflection, i.e., roots of the second derivative of F, found via Newton's method. These ensure that each subdomain admits a monotonic F', enabling the local invertibility required for $\mathcal{N}_G^{(k)}$. The training of the $k+1$ networks is performed by minimizing the MSE loss $\varepsilon(\tilde{u}^{(s)}, u^{(s)})$, where $\tilde{u}^{(s)} = \mathcal{N}_G^{(k)}(\delta\tilde{y}^{(s)})$. The rarefaction wave is then reconstructed by replacing F' with $\mathcal{N}_F'$ and G with the corresponding $\mathcal{N}_G^{(k)}$ in the right-hand expression of (3). By combining the ANN-based shock and rarefaction approximations, the ANN-Flux method is capable of solving Riemann

[1]marcelomokwademello@gmail.com
[2]arthurmes@ufrgs.br
[3]pedro.konzen@ufrgs.br

problems involving nonconvex flux functions. The Buckley–Leverett equation models two-phase immiscible and incompressible fluid flow through porous media [5]. Its flux function is nonconvex, exhibiting a single change in concavity within the admissible domain of the variable. To solve the equation, it is necessary to determine whether the left and right states of the solution lie within the same concavity region of the flux function. If both states fall within the same region, the solution consists of either a shock or a rarefaction wave. However, if the states lie in different concavity regions, a convex (or concave) envelope must be constructed to connect them. In the Buckley–Leverett case, this construction corresponds to using the lower convex envelope when $u_L < u_R$, or the upper concave envelope when $u_R < u_L$, effectively redefining the flux function accordingly. As a preliminary result, we compared the ANN-F solution with a reference numerical solution of the Buckley–Leverett equation. The numerical inverse function G was locally computed using Newton's method. The MLP architectures were selected based on numerical experiments, from which we concluded that a 1–50 × 3–1 network for $\mathcal{N}_F$ and a 1–50 × 4–1 network for $\mathcal{N}_G$ were suitable for approximating F and G, respectively, achieving an L_2 error of 10^{-6}. Training was performed using hyperbolic tangent activation functions in the hidden layers and the identity function in the output layer, with the mean squared error as the loss function.The ANN-F method yielded accurate results for the Buckley–Leverett equation, with a relative L_2 error of 10^{-4}. Future work will focus on extending the method to more intricate conservation laws, such as those arising in traffic flow modeling [1].

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